From localization of simple groups to nilpotency

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joint work with José Cantarero and Antonio Viruel

1990, Jacques' 40th birthday

1. **Jacques a dit**: Le cardinal de l'orbite est égal à l'indice du stabilisateur.

Homotopical localization

The work of Bousfield and Dror Farjoun had a deep impact in homotopy theory.

If $f : A \to B$ is a map between two spaces, the *localization functor* L_f is a coaugmented and idempotent functor from spaces to spaces which "inverts f".

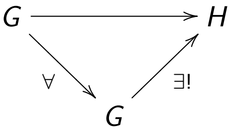
Example

1. Localization at a prime: $X \to X_{(p)}$;
2. Completion at a prime: $X \to X_p^\wedge$;
3. Postnikov sections such as $X \to X[1] = B\pi_1 X$;
4. Quillen's plus construction $X \to X^+$;
5. $X \to L_{B\mathbb{Z}/p}X$.

Localization of groups

With José Luis Rodríguez we started looking at localization functors in the category of groups, trying to understand what properties are preserved by localization.

Instead of looking at the whole localization functor L, it is good enough to consider $\alpha : G \to LG$ because $L_\alpha G \cong LG$. To check if H can be obtained from G by a localization functor, one needs to verify a universal property:



Libman had found examples such as $A_{10} \to A_{11}$ which show that new torsion can be created.

2000, Jacques' 50th birthday

Definition

Two finite simple groups lie in the same *localization component* if they can be connected by a zigzag of localization.

Theorem

Any sporadic simple group, except possibly the Monster, lies in the same localization component as any alternating group.

Question

Is there a single localization component?

Preservation of nilpotency?

Theorem (Libman 2000/Aschbacher 2004)

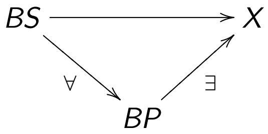
The localization of any nilpotent group of class $c \leq 3$ is again nilpotent of class ≤ 3.

With Antonio Viruel we used a very subtle (and incorrect) induction to prove it for $c \geq 4$.

Consider a finite p-group S and a map $f : BS \to X$ such that:

1. $H^1(X; \mathbb{F}_p) \twoheadrightarrow H^1(BS; \mathbb{F}_p)$ is an epimorphism;
2. $H^2(X; \mathbb{F}_p) \hookrightarrow H^2(BS; \mathbb{F}_p)$ is a monomorphism.

Then f enjoy a "semi-localization" property:



Stammbach's criterion

Definition (Kessar-Linckelmann)

A p-local finite group is *nilpotent* if its fusion system $\mathcal{F}$ is that of the Sylow subgroup S.

This happens if and only if $|\mathcal{L}|^\wedge_p \simeq BS$, or even if and only if BS is a retract of $|\mathcal{L}|^\wedge_p$.

Theorem

(2005?) A p-local finite group is nilpotent if and only if $H^1(|\mathcal{L}|^\wedge_p; \mathbb{F}_p) \cong H^1(BS; \mathbb{F}_p)$.

This is straightforward from the "semi-localization criterion".

Jacques' 60th birthday

In 2009 José Cantarero joins and we finally go through our "alphabetical theorem". In particular:

Theorem (Frobenius criterion)

A p-local finite group is nilpotent if and only if $\mathrm{Aut}_{\mathcal{F}}(P)$ *is a p-group for every* $P \leq S$.

Also present in work of Linckelmann.

Atiyah-Quillen criterion

Theorem

A p-local finite group is nilpotent if and only if $H^n(|\mathcal{L}|_p^\wedge; \mathbb{F}_p) \cong H^n(BS; \mathbb{F}_p)$ *is an isomorphism for* $n >> 0$.

Related to a recent result of Díaz, Glesser, Park, and Stancu about a result of Tate for fusion systems.

Castellana and Morales generalized the result of Hopkins, Kuhn, and Ravenel on generalized characters of finite groups to the setting of p-local finite groups.

Theorem

A p-local finite group is nilpotent if and only if $K(n)^*(|\mathcal{L}|_p^\wedge) \cong K(n)^*(BS)$ *is an isomorphism.*