Applications of Characteristic Elements for Saturated Fusion Systems

Sejong Park
Joint work in part with Kári Ragnarsson

University of Aberdeen

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- $\mathcal{F}$ saturated fusion system on a finite p-group S
- (S, S)-biset $\rightsquigarrow$ the cohomology ring of $\mathcal{F}$ (Linckelmann-Webb, Broto-Levi-Oliver)
- Example: For $\mathcal{F} = \mathcal{F}_S(G)$, G finite group, $S \in \mathrm{Syl}_p(G)$, the (S, S)-biset decomposition

$$ {}_S G_S = \coprod_{x \in [S \backslash G / S]} SxS \cong \coprod_{x \in [S \backslash G / S]} S \times_{(S \cap S^x, c_x)} S $$

"determines" the transfer map $H^*(S, \mathbb{F}_p) \xrightarrow{\mathrm{tr}_S^G} H^*(G, \mathbb{F}_p)$ by Mackey decomposition formula:

$$ \mathrm{res}_S^G \circ \mathrm{tr}_S^G = \sum_{x \in [S \backslash G / S]} \mathrm{tr}_{S \cap S^x}^S \circ c_x^* \circ \mathrm{res}_{S \cap S^x}^S . $$

Bisets

- S_1, S_2 groups
- (S_1, S_2)-biset $=$ a set X with compatible right S_1-action and left S_2-action
- We'll consider only those in which both left and right actions are free (bifree).
- Transitive (S_1, S_2)-bisets: for $P \leq S_1$, $\varphi\colon P \to S_2$ injective,

$$S_2 \times_{(P,\varphi)} S_1 := S_2 \times S_1 / \sim, \quad (y\varphi(u), x) \sim (y, ux) \quad \text{for } x \in S_1, y \in S_2, u \in P$$
$$t \cdot [y, x] \cdot s = [ty, xs] \qquad \text{for } x, s \in S_1, y, t \in S_2$$

- An (S_1, S_2)-bisets X can be viewed as a (left) $(S_2 \times S_1)$-set via $(t, s) \cdot x = txs^{-1}$.
- Transitive $(S_2 \times S_1)$-sets: for $P \leq S_1$, $\varphi\colon P \to S_2$ injective

$$S_2 \times S_1 / \Delta_P^{\varphi}, \quad \Delta_P^{\varphi} = \{(\varphi(x), x) \mid x \in P\}$$

- $[X]$ = the isomorphism class of (S_1, S_2)-bisets containing X
- $[P, \varphi]_{S_1}^{S_2} := [S_2 \times_{(P,\varphi)} S_1]$

Double Burnside Ring

- $A(S_1, S_2)$ = the Grothendieck group of isomorphism classes of (S_1, S_2)-bisets = the free abelian group on the basis elements $[P,\varphi]_{S_1}^{S_2}$
- $\circ\colon A(S_2, S_3) \times A(S_1, S_2) \to A(S_1, S_3), \quad [Y] \circ [X] = [Y \times_{S_2} X]$

$$[Q,\psi]_{S_2}^{S_3} \circ [P,\varphi]_{S_1}^{S_2} = \sum_{x \in [Q \backslash S_2 / \varphi(P)]} [\varphi^{-1}(\varphi(P) \cap Q^x), \psi \circ c_x \circ \varphi]_{S_1}^{S_3}$$

- $\epsilon\colon A(S_1, S_2) \to \mathbb{Z}, \quad \epsilon([X]) = |S_2 \backslash X| = |X|/|S_2|$

$$\epsilon([Y] \circ [X]) = \epsilon([Y])\epsilon([X])$$

- $c_{[P,\varphi]}\colon A(S_1, S_2) \to \mathbb{Z}$ the coefficient function for $[P,\varphi]$.
- $A(S,S)$ = double Burnside ring of the group S (wrt the product $\circ$)
- $A^+(S,S) = \{\omega \in A(S,S) \mid c_{[P,\varphi]}(\omega) \geq 0\}$
- $A(S,S)_{(p)} = A(S,S) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$

Stable Elements

- $\mathcal{F}$ fusion system on a finite p-group S
- $A_{\mathcal{F}}(S,S) = \mathbb{Z}$-span of $\{[P,\varphi]_S^S \mid P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\}$
- $A_{\mathcal{F}}(S,S)$ is a subring of $A(S,S)$.
- An element $\omega \in A(S,S)$ is called right $\mathcal{F}$-stable if

$$\omega \circ [P,\varphi]_P^S = \omega \circ [P,\mathrm{incl}]_P^S \quad \text{whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S);$$

  similarly $\omega \in A(S,S)$ is called left $\mathcal{F}$-stable if

$$[\varphi(P),\varphi^{-1}]_S^P \circ \omega = [P,\mathrm{id}]_S^P \circ \omega \quad \text{whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S).$$

- If $\omega = [X] \in A^+(S,S)$, then ω is right $\mathcal{F}$-stable if

$$X|_{(S,\varphi)} \cong X|_{(S,P)} \text{ as } (S,P)\text{-bisets whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S);$$

  ω is left $\mathcal{F}$-stable if

$$X|_{(\varphi,S)} \cong X|_{(P,S)} \text{ as } (P,S)\text{-bisets whenever } P \le S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S).$$

Theorem (Broto-Levi-Oliver)

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there are elements $\omega \in A_{\mathcal{F}}^{+}(S,S)$ with $\epsilon(\omega) \not\equiv 0 \mod p$ which are both left and right $\mathcal{F}$-stable.

We call such an ω a characteristic biset for $\mathcal{F}$.

Example

Let $\mathcal{F} = \mathcal{F}_S(G), S \in \mathrm{Syl}_p(G)$. Then ${}_S G_S$ is a characteristic biset for $\mathcal{F}_S(G)$ and

$$[{}_S G_S] = \sum_{x \in [S \backslash G / S]} [S \cap S^x, c_x]_S^S$$

Problem: Characteristic bisets are not unique.

Theorem (Ragnarsson)

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there is a unique idempotent $\omega_{\mathcal{F}} \in A_{\mathcal{F}}(S,S)_{(p)}$ with $\epsilon(\omega_{\mathcal{F}}) = 1$ that is both left and right $\mathcal{F}$-stable.

We call $\omega_{\mathcal{F}}$ the characteristic idempotent of $\mathcal{F}$.

Characteristic Idempotents determine saturated fusion systems

Theorem (Ragnarsson-Stancu)

Let S be a finite p-group. Then there is a bijection

$$\{\mathcal{F} \mid \mathcal{F} \text{ sfs on } S\} \leftrightarrow \{\omega \in A(S,S)_{(p)} \mid \omega^2 = \omega, \epsilon(\omega) = 1,\ \omega \text{ s.t. Frobenius reciprocity}\}$$

$$\mathcal{F} \mapsto \omega_{\mathcal{F}}$$

$$\mathrm{RSt}(\omega) \leftarrow\!\shortmid\ \omega$$

- $\omega \in A(S,S)_{(p)}$ satisfies Frobenius reciprocity if

$$(\omega \times \omega) \circ [S,\Delta]_S^{S\times S} = (\omega \times S) \circ [S,\Delta]_S^{S\times S} \circ \omega \in A(S, S\times S)_{(p)}.$$

- $\mathrm{RSt}(\omega)$ is the category whose objects are the subgroups of S and such that for all $P, Q \leq S$,

$$\mathrm{Hom}_{\mathrm{RSt}(\omega)}(P,Q) = \{\varphi \in \mathrm{Inj}(P,Q) \mid \omega \circ [P,\varphi]_P^S = \omega \circ [P,\mathrm{incl}]_P^S\}$$

Theorem (Puig)

$\mathrm{RSt}(\omega_{\mathcal{F}}) = \mathcal{F}$.

Definition

Let S be a p-subgroup of G. Define $\mathcal{F} = \mathcal{F}_S(G)$ to be the category with

- objects: all subgroups of S
- morphisms: $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_G(P, Q)$ for $P, Q \leq S$

Proposition

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then there is a finite group G containing S as a subgroup such that $\mathcal{F} = \mathcal{F}_S(G)$.

Proof.

Let $[X] \in A^{+}_{\mathcal{F}}(S, S)$ be right $\mathcal{F}$-stable with $e = |X|/|S| \not\equiv 0 \mod p$. Then

$$\begin{aligned} \iota\colon S &\to G := \mathrm{Aut}(X|_{(1,S)}) \cong S \wr \Sigma_e \\ u &\mapsto (x \mapsto ux) \end{aligned}$$

is injective, and identifying S with $\iota(S)$, we have $\mathcal{F} = \mathcal{F}_S(G)$. □

Definition

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group T. Define the exoticity index of $\mathcal{F}$ to be the number

$$\min\{\log_p |S : T| \colon T \leq S \in \mathrm{Syl}_p(G) \text{ for some finite group } G \text{ s.t. } \mathcal{F} = \mathcal{F}_T(G)\}.$$

Proposition

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then $\mathcal{F}$ is exotic iff it has positive exoticity index.

Question

Given an exotic fusion system $\mathcal{F}$ on a finite p-group S, what is the exoticity index of $\mathcal{F}$, and what are the finite groups G achieving the exoticity index?

Question

Given an exotic fusion system $\mathcal{F}$ on a finite p-group S, what is the smallest upper bound of the exoticity index of $\mathcal{F}$ which can be obtained from the biset construction?

- $\mathcal{F}$ a saturated fusion system on a finite p-group S
- $\omega = \omega_{\mathcal{F}}$ the characteristic idempotent of $\mathcal{F}$
- Write

$$\omega = \omega_0 + \omega_1 + \omega_2 \cdots ,$$

where

$$\omega_r = \sum_{\substack{P \leq S \\ |S:P| = p^r}} \sum_{[P,\varphi]} c_{[P,\varphi]}(\omega)[P,\varphi]_S^S.$$

Proposition

We have

$$\omega_0 = c_0 \sum_{[\alpha] \in \mathrm{Out}_{\mathcal{F}}(S)} [S,\alpha]_S^S,$$

where $\mathrm{Out}_{\mathcal{F}}(S) = \mathrm{Aut}_{\mathcal{F}}(S)/\mathrm{Aut}_S(S)$ *and* $c_0 = \frac{1}{|\mathrm{Out}_{\mathcal{F}}(S)|}$. *In fact,*

$$\omega_0 = \omega_{N_{\mathcal{F}}(S)}.$$

Proposition

$$\omega_1 = \sum_{\varphi_{i,j} \text{ extendable}} c_i^e [P_i, \varphi_{i,j}]_S^S + \sum_{\varphi_{i,j} \text{ nonextendable}} c_i^n [P_i, \varphi_{i,j}]_S^S,$$

where

$$c_i^e = -\frac{d_i^n}{d_i^e + p d_i^n} c_0, \quad c_i^n = \frac{d_i^e}{d_i^e + p d_i^n} c_0.$$

- Write
$$\omega_1 = \sum_{i=1}^{n} \sum_{j=1}^{m_i} c_{i,j} [P_i, \varphi_{i,j}]_S^S$$
- $\varphi_{i,j}$ is *extendable* (in $\mathcal{F}$) if there is an $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ s.t. $\alpha|_{P_i} = \varphi_{i,j}$
- $\varphi_{i,j}$ is *nonextendable* otherwise.
- $d_i^e = |\{1 \leq j \leq m_i \mid \varphi_{i,j} \text{ is extendable}\}|$
- $d_i^n = |\{1 \leq j \leq m_i \mid \varphi_{i,j} \text{ is nonextendable}\}|$

Proposition

Let $\mathcal{F}$ be a saturated fusion system on $S \cong p_+^{1+2}$, p odd, such that all rank two elementary abelian subgroups V_i are $\mathcal{F}$-radical. Then there is a unique minimal right $\mathcal{F}$-stable $[X] \in A_{\mathcal{F}}^{+}(S,S)$ with $|X|/|S| \not\equiv 0 \mod p$.

p	$\mathrm{Out}_{\mathcal{F}}(S)$	$\lvert V_i^{\mathcal{F}} \rvert$	$\mathrm{Out}_{\mathcal{F}}(V_i)$	Group	e
3	D_8	$2+2$	$\mathrm{GL}_2(3)$	${}^2F_4(2)'$	$8 \cdot 121$
3	SD_{16}	4	$\mathrm{GL}_2(3)$	J_4	$16 \cdot 121$
5	$4S_4$	6	$\mathrm{GL}_2(5)$	Th	$96 \cdot 781$
7	$6^2 : 2$	$6+2$	$\mathrm{SL}_2(7):2$, $\mathrm{GL}_2(7)$		$72 \cdot 2801$
7	$D_{16} \times 3$	$4+4$	$\mathrm{SL}_2(7):2$		$48 \cdot 2801$
7	$SD_{32} \times 3$	8	$\mathrm{SL}_2(7):2$		$96 \cdot 2801$

$$e = |X|/|S| = \frac{p^5-1}{p-1}|\mathrm{Out}_{\mathcal{F}}(S)|$$

- Question: Relation between $\omega_{\mathcal{F}}$ and $\omega_{O^p(\mathcal{F})}$?
- Group case: G finite group, $S \in \mathrm{Syl}_p(G)$
- $N := O^p(G) \Rightarrow G = SN \Rightarrow G|_{(S,N)} \cong S \times_L N$ where $L := S \cap N$
- $\mathcal{F} := \mathcal{F}_S(G)$, $L = \mathrm{hyp}(\mathcal{F})$, $\mathcal{F}_L(N) = O^p(\mathcal{F})$.

Theorem (Ragnarsson)

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Then

$$\omega_{\mathcal{F}} \circ [T, \mathrm{incl}]_T^S = [T, \mathrm{incl}]_T^S \circ \omega_{O^p(\mathcal{F})} \qquad \text{where } T = \mathrm{hyp}(\mathcal{F}).$$

Theorem (Tate; Gagola-Isaacs; Díaz-Glesser-P-Stancu)

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\mathcal{H}$ be a saturated subsystem of $\mathcal{F}$ on S. Then

$$\mathrm{foc}(\mathcal{F}) = \mathrm{foc}(\mathcal{H}) \iff \mathrm{hyp}(\mathcal{F}) = \mathrm{hyp}(\mathcal{H}).$$

Generalize?

Definition

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $\mathcal{H}$, $\mathcal{K}$ be saturated subsystems of $\mathcal{F}$ on $R, T \leq S$, resp. Write ${}_R\mathcal{F}_T = \mathcal{HK}$ if for every $P \leq T$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, R)$, we have $\varphi = \eta \circ \kappa$ for some η in $\mathcal{H}$ and κ in $\mathcal{K}$.

Conjecture

Suppose ${}_R\mathcal{F}_T = \mathcal{HK}$*. Then we have*

$$[R, \mathrm{id}]_S^R \circ \omega_{\mathcal{F}} \circ [T, incl]_T^S = \omega_{\mathcal{H}} \circ [R, \mathrm{id}]_S^R \circ [T, incl]_T^S \circ \omega_{\mathcal{K}}\,.$$

Proposition (Ragnarsson-P)

Let $(\mathcal{F}, S) \geq (\mathcal{H}, S), (\mathcal{K}, T)$*. Suppose* ${}_S\mathcal{F}_T = \mathcal{HK}$*. Then Conjecture holds if* $\mathcal{K}$ *is weakly normal in* $\mathcal{F}$*. In particular,* $\omega_{\mathcal{F}} = \omega_{N_{\mathcal{F}}(S)} \circ \omega_{O^{p'}(\mathcal{F})}$.

Question

How about $\mathcal{F} = \langle N_{\mathcal{F}}(S), N_{\mathcal{F}}(Q)\rangle$*?*