Exotic fusion systems over 2-groups

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The **fusion category** of a finite group G encodes the conjugacy relations within a Sylow p-subgroup S of G.

For any group G and any pair of subgroups $H, K \leq G$, set

$$\mathrm{Hom}_G(H,K) = \Big\{\varphi \in \mathrm{Hom}(H,K) \;\Big|\; \varphi = (x \mapsto gxg^{-1}),\ \text{some } g \in G\Big\}.$$

When G is a finite group and $S \in \mathrm{Syl}_p(G)$, let $\boldsymbol{\mathcal{F}_S(G)}$ be the category where

$$\mathrm{Ob}(\mathcal{F}_S(G)) = \{P \leq S\}$$
$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q).$$

The notion of an abstract fusion system is due to Puig. The definitions we give here are modified, but equivalent versions of Puig's definitions.

Definition *A* **fusion system** *over a p-group S is a category $\mathcal{F}$, where* Ob($\mathcal{F}$) *is the set of all subgroups of S, and which satisfies the following two properties for all $P, Q \leq S$:*

- $\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q)$*; and*
- *each $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P,Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.*

By analogy with the terminology for finite groups, two subgroups $P, Q \leq S$ are **$\mathcal{F}$-conjugate** if they are isomorphic as objects of the category $\mathcal{F}$.

Definition (Roberts & Shpectorov) *Let $\mathcal{F}$ be a fusion system over a p-group S.*

- *A subgroup $P \leq S$ is* **fully automised** *in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.*
- *A subgroup $P \leq S$ is* **receptive** *in $\mathcal{F}$ if it has the following property: for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set*

$$N_\varphi = \Big\{ g \in N_S(Q) \,\Big|\, \exists\, h \in N_S(P) \text{ with } \varphi c_g \varphi^{-1} = c_h \in \mathrm{Aut}(P) \Big\},$$

  then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.
- *A fusion system $\mathcal{F}$ over a p-group S is* **saturated** *if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automised and receptive.*

The fusion category $\mathcal{F}_S(G)$ of a finite group G is clearly a fusion system. It is not hard to show, using the Sylow theorems, that it also satisfies the saturation conditions.

A saturated fusion system $\mathcal{F}$ over a p-group S is called **realizable** if it is equal to $\mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and is called **exotic** otherwise.

When p is odd, we know many examples of saturated fusion systems over p-groups which are exotic.

When $p = 2$, we know one family of exotic fusion systems, over groups of order 2^{10+3k} for all $k \geq 0$. These were predicted to exist by Solomon and Benson, and first constructed by Levi and Oliver.

The Solomon fusion systems, and others easily constructed from them, are the only exotic saturated fusion systems known to exist over 2-groups.

We now look at some of the basic definitions for subgroups in fusion systems and types of fusion systems.

Definition *Fix a prime p, a finite p-group S, and a fusion system $\mathcal{F}$ over S. Let $P \leq S$ be any subgroup.*

- *P is* **central** *in $\mathcal{F}$ if $P \lhd S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar{\varphi}|_P = \mathrm{Id}_P$.*
  $\boldsymbol{Z(\mathcal{F})} \leq S$ is the largest subgroup of S which is central in $\mathcal{F}$.

- *P is* **normal** *in $\mathcal{F}$ ($P \lhd \mathcal{F}$) if $P \lhd S$, and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ in $\mathcal{F}$ extends to a morphism $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PQ,PR)$ such that $\bar{\varphi}(P) = P$.*
  $\boldsymbol{O_p(\mathcal{F})} \leq S$ is the largest subgroup of S which is normal in $\mathcal{F}$.

- *P is* **strongly closed** *in $\mathcal{F}$ if no element of P is $\mathcal{F}$-conjugate to an element of $S \smallsetminus P$.*

Several definitions of normal or weakly normal fusion subsystems have been given.

Definition *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S, and let $\mathcal{E} \subseteq \mathcal{F}$ be a saturated fusion subsystem over $T \leq S$. Then $\mathcal{E}$ is* **weakly normal** *in $\mathcal{F}$ ($\mathcal{E} \lhd \mathcal{F}$) if*

(a) T is strongly closed in $\mathcal{F}$;

(b) for each $P \leq T$ and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,T)$, there are $\alpha \in \mathrm{Aut}_{\mathcal{F}}(T)$ and $\varphi_0 \in \mathrm{Hom}_{\mathcal{E}}(P,T)$ such that $\varphi = \alpha \circ \varphi_0$; and

(c) for each $P, Q \leq T$, each $\varphi \in \mathrm{Hom}_{\mathcal{E}}(P,Q)$, and each $\beta \in \mathrm{Aut}_{\mathcal{F}}(T)$,

$$^{\beta}\varphi \in \mathrm{Hom}_{\mathcal{E}}(\beta(P), \beta(Q)).$$

A saturated fusion system $\mathcal{F}$ is **simple** if it contains no nontrivial proper weakly normal subsystems.

By a theorem of David Craven, a saturated fusion system is simple under the above definition if and only if it has no nontrivial proper *normal* subsystems, as defined by Aschbacher.

When G is a finite group, $\boldsymbol{O^p(G)}$ and $\boldsymbol{O^{p'}(G)}$ are defined to be the smallest normal subgroups of p-power index, and of index prime to p, respectively. We need to look at analogous subsystems of saturated fusion systems.

For any saturated fusion system $\mathcal{F}$, the **focal subgroup** $\mathfrak{foc}(\mathcal{F})$ and the **hyperfocal subgroup** $\mathfrak{hyp}(\mathcal{F})$ are defined by

$$\begin{aligned}\boldsymbol{\mathfrak{foc}(\mathcal{F})} &= \Big\langle s^{-1}t \,\Big|\, s,t \in S \text{ are } \mathcal{F}\text{-conjugate} \Big\rangle \\ &= \Big\langle s^{-1}\alpha(s) \,\Big|\, s \in P \le S,\ \alpha \in \mathrm{Aut}_{\mathcal{F}}(P) \Big\rangle\end{aligned}$$

$$\boldsymbol{\mathfrak{hyp}(\mathcal{F})} = \Big\langle s^{-1}\alpha(s) \,\Big|\, s \in P \le S,\ \alpha \in O^p(\mathrm{Aut}_{\mathcal{F}}(P)) \Big\rangle.$$

When $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G and some $S \in \mathrm{Syl}_p(G)$, then

$$\mathfrak{foc}(\mathcal{F}) = S \cap [G,G]$$

by the **focal subgroup theorem**, and

$$\mathfrak{hyp}(\mathcal{F}) = S \cap O^p(G)$$

by the **hyperfocal subgroup theorem** of Puig.

The existence of saturated fusion subsystems analogous to the subgroups $O^p(G)$ and $O^{p'}(G)$ was shown by Broto, Castellana, Grodal, Levi, & Oliver; and (in part) independently by Puig.

Theorem *For each saturated fusion system $\mathcal{F}$ over a finite p-group S, there is a unique saturated fusion subsystem $\boldsymbol{O^p(\mathcal{F})}$ over $\mathfrak{hyp}(\mathcal{F})$ such that for each $P \le \mathfrak{hyp}(\mathcal{F})$,*

$$\mathrm{Aut}_{O^p(\mathcal{F})}(P) \ge O^p(\mathrm{Aut}_{\mathcal{F}}(P)).$$

In fact, $O^p(\mathcal{F})$ is weakly normal in $\mathcal{F}$, and is the fusion subsystem of $\mathcal{F}$ generated by $\mathrm{Inn}(\mathfrak{hyp}(\mathcal{F}))$ and the groups $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for $P \le \mathfrak{hyp}(\mathcal{F})$.

Theorem *For each saturated fusion system $\mathcal{F}$ over a finite p-group S, there is a unique minimal weakly normal fusion subsystem $\boldsymbol{O^{p'}(\mathcal{F})}$ over S. This has the property that for each $P \le S$,*

$$\mathrm{Aut}_{O^{p'}(\mathcal{F})}(P) \ge O^{p'}(\mathrm{Aut}_{\mathcal{F}}(P)).$$

Definition *A* **reduced fusion system** *is a saturated fusion system* $\mathcal{F}$ *such that*

- $\mathcal{F}$ *has no nontrivial normal* p*-subgroup,*
- $\mathcal{F}$ *has no proper normal subsystem of* p*-power index, and*
- $\mathcal{F}$ *has no proper normal subsystem of index prime to* p*.*

Equivalently, $\mathcal{F}$ *is reduced if* $O_p(\mathcal{F}) = 1$, $O^p(\mathcal{F}) = \mathcal{F}$, *and* $O^{p'}(\mathcal{F}) = \mathcal{F}$.

For any saturated fusion system $\mathcal{F}$, the **reduction** of $\mathcal{F}$ is defined by the following procedure:

- set $Q = O_p(\mathcal{F})$ and $\mathcal{F}_0 = C_{\mathcal{F}}(Q)/Z(Q)$;
- let $\mathcal{F}_0 \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_m$ be such that $\mathcal{F}_i = O^p(\mathcal{F}_{i-1})$ if i is odd, $\mathcal{F}_i = O^{p'}(\mathcal{F}_{i-1})$ if i is even, and $O^p(\mathcal{F}_m) = O^{p'}(\mathcal{F}_m) = \mathcal{F}_m$; and
- set $\mathfrak{red}(\mathcal{F}) = \mathcal{F}_m$.

As one might expect for the definition to make sense, $\mathfrak{red}(\mathcal{F})$ is reduced for any saturated fusion system $\mathcal{F}$.

Definition *A saturated fusion system $\mathcal{F}$ over a p-group S is* **tame** *if it is realized by a finite group G with $S \in \mathrm{Syl}_p(G)$, which also has the property that the natural homomorphism*

$$\mathrm{Out}(G) \longrightarrow \mathrm{Out}(BG^\wedge_p)$$

is split surjective.

In particular, each tame fusion system is realizable. It seems likely that there should be realizable fusion systems which are not tame, but we have not found any examples.

Note: $BG^\wedge_p$ is the p-completion of the classifying space BG of G, and $\mathrm{Out}(BG^\wedge_p)$ is the group of homotopy classes of self homotopy equivalences of this space.

By a theorem of Broto-Levi-Oliver, $\mathrm{Out}(BG^\wedge_p)$ can be described purely algebraically, as a certain group of outer automorphisms of the linking category of G.

Here are our main theorems [AOV] on reduced and tame fusion systems:

Theorem A *For any saturated fusion system* $\mathcal{F}$ *over a finite* p*-group* S*, if* $\mathfrak{red}(\mathcal{F})$ *is tame, then* $\mathcal{F}$ *is also tame, and in particular* $\mathcal{F}$ *is realizable.*

Theorem B *Let* $\mathcal{F}$ *be a reduced fusion system which is not tame. Then there is an exotic fusion system whose reduction is isomorphic to* $\mathcal{F}$*.*

Reduced fusion systems can be very far from being simple in any sense. For example, a product of reduced fusion systems is always reduced. The next theorem handles reduced fusion systems which factor as products.

Theorem C *Each reduced fusion system* $\mathcal{F}$ *over a finite* p*-group* S *has a unique maximal factorization* $\mathcal{F} = \mathcal{F}_1 \times \cdots \times \mathcal{F}_m$ *as a product of reduced indecomposable fusion systems* $\mathcal{F}_i$ *over subgroups* $S_i \lhd S$*. If* $\mathcal{F}_i$ *is tame for each* i*, then* $\mathcal{F}$ *is tame.*

Theorems A, B, and C reduce the search for exotic fusion systems to a search for reduced, indecomposable fusion systems which are not tame.

Of course, reduced, indecomposable fusion systems need not be simple. For example, the fusion system at 2 of the wreath product $A_6 \wr A_5$ is reduced and indecomposable, but contains the fusion system of $(A_6)^5$ as proper normal subsystem. This fusion system is over a group of order 2^{17}, and we do not know any example over a smaller 2-group.

One can make a list of conditions on a finite 2-group S which are necessary for there to exist any reduced fusion systems over S: conditions which are explicit enough that they can be checked by computer.

Using MAGMA and GAP and their database of 2-groups, we were able to list all reduced, indecomposable fusion systems over 2-groups of order $\leq 2^9$, and prove that they are all tame. Moreover, they are all realized by finite simple groups. Together with Theorem A, this proves that every saturated fusion system over a 2-group of order $\leq 2^9$ is realizable. (The smallest known exotic fusion system at 2 is over a group of order 2^{10}.)

For example, these formal criteria allowed us to eliminate all but 34 of the 10 million groups of order 2^9. Of those 34:

- 10 are Sylow 2-subgroups of simple groups
- 11 contain a nontrivial abelian direct factor
- 12 others are products of two nonabelian groups
- 1 does not fit in any of the above classes.

This forced us to give special consideration to reduced fusion systems over 2-groups which factor nontrivially as products.

Theorem 1 *Fix 2-groups S_1 and S_2, and set $S = S_1 \times S_2$. Assume, for $i = 1, 2$,*

(a) $\Omega_1(Z(S_i)) \leq [S_i, S_i]$, *and* $\forall\ \varphi \in \mathrm{Hom}(S_i, S_i)$ *such that* $\varphi(S_i) \lhd S_i$ *and* φ *is not onto,* $\mathrm{Ker}(\varphi) \geq \Omega_1(Z(S_i))$*; and*

(b) S_i *contains no subgroup isomorphic to* $S_{3-i} \times S_{3-i}$.

Then for every saturated fusion system $\mathcal{F}$ over S such that $O^2(\mathcal{F}) = \mathcal{F} = O^{2'}(\mathcal{F})$, $\mathcal{F} \cong \mathcal{F}_1 \times \mathcal{F}_2$ for some pair of saturated fusion systems $\mathcal{F}_i$ over S_i.

There is a second theorem, with the same conclusion, where one assumes S_1 is dihedral, semidihedral, or $C_{2^n} \wr C_2$, but where the only condition on S_2 is that it not contain $S_1 \times S_1$.

Assumption (b) is definitely needed. For example, $D_8 \times (D_8 \wr C_2)$ is a Sylow 2-subgroup of A_{14}, whose fusion system is reduced and indecomposable. Similar examples involving fusion systems of certain linear or orthogonal groups show that when S is any dihedral or semidihedral 2-group, $S \times (S \wr C_2)$ supports a reduced, indecomposable fusion system.

The following result is useful when working with fusion systems over 2-groups which contain an abelian direct factor.

Theorem 2 *Fix a 2-group S. Set*

$$S_0 = \Omega_1(Z(S)) \quad \text{and} \quad S_1 = S_0 \cap [S,S].$$

- *If $[S_0 : S_1] = 2$, or if $S_0 \gneqq S_1$ and $\mathrm{Out}(S)$ is a 2-group, then $O^2(\mathcal{F}) \subsetneqq \mathcal{F}$ for every saturated fusion system $\mathcal{F}$ over S.*
- *More generally, if $S_0 \gneqq S_1$, $\mathcal{F}$ is a saturated fusion system over S and $O^2(\mathcal{F}) = \mathcal{F}$, then the action of $\mathrm{Out}_{\mathcal{F}}(S)$ on $Z(S)$ induces an action on S_0/S_1 with trivial fixed component.*

The proof of this theorem is based on the observation that

$$O^p(\mathcal{F}) = \mathcal{F} \iff \mathfrak{hyp}(\mathcal{F}) = S \iff \mathfrak{foc}(\mathcal{F}) = S$$

for any saturated fusion system $\mathcal{F}$ over a p-group S, together with a **transfer homomorphism**

$$\mathrm{trf}_{\mathcal{F}} \colon S/\mathfrak{foc}(S) \xrightarrow{\;1-1\;} S/[S,S] \ .$$

If $g \in \Omega_1(Z(S))$ is fixed by the action of $\mathrm{Out}_{\mathcal{F}}(S)$, then $\mathrm{trf}_{\mathcal{F}}([g]) = [g]$.

Theorem 2 leads us to consider the following class of nonabelian 2-groups:

$$\mathcal{C} = \Big\{ S \text{ a nonabelian 2-group} \,\Big|\, \Omega_1(Z(S)) \nleq [S,S] \Big\}.$$

Every nonabelian 2-group with abelian direct factor is in $\mathcal{C}$, but this class also contains other groups, such as $C_4 \rtimes C_4$.

The next proposition is a straightforward (and tedious) application of the classification theorem for finite simple groups.

Proposition 3 *No finite simple group has Sylow 2-subgroups in the class $\mathcal{C}$.*

This makes it natural to conjecture:

Conjecture 4 *Each reduced fusion system over a finite 2-group $S \in \mathcal{C}$ is exotic.*

However, Conjecture 4 has the following counterexample.

Example 5 *Consider the group*

$$G = \mathrm{Ker}\Big[(\Sigma_6)^{15} \rtimes A_5 \twoheadrightarrow C_2^{15} \rtimes A_5 \twoheadrightarrow C_2 \times A_5 \xrightarrow{\mathrm{pr}_1} C_2\Big].$$

Here, A_5 permutes transitively the 15 factors Σ_6. Fix $S \in \mathrm{Syl}_2(G)$, and set $\mathcal{F} = \mathcal{F}_S(G)$. Then $\mathcal{F}$ is a reduced fusion system.

Since Σ_6 has Sylow 2-subgroups $\cong D_8 \times C_2$,

$$S \cong \Big((D_8)^{15} \times C_2^{14}\Big) \rtimes C_2^2$$

and contains a direct factor C_2^2.

Thus $|S| = 2^{61}$, $S \in \mathcal{C}$, and $\mathcal{F}$ is a counterexample to Conjecture 4.

With a little more work, one can modify Example 5 to get a realizable, reduced fusion system over a group $S \in \mathcal{C}$ of order 2^{57}.

In fact, with the help of Proposition 3, and additional applications of the classification of finite simple groups, we can show the following:

Proposition 6 *Let $\mathcal{F}$ be a reduced fusion system over a finite 2-group $S \in \mathcal{C}$. Then either $\mathcal{F}$ is exotic, or $|S| \geq 2^{57}$.*

It seems very hard to prove any general results on the nonexistence of reduced fusion systems over groups in the class $\mathcal{C}$. The following is the most general such result we know.

Proposition 7 *Let S_0 be a 2-group such that*

- $\mathrm{rk}(S_0) \leq 5$ *(i.e., $C_2^6 \not\leq S_0$),*
- $\Omega_1(Z(S_0)) \leq \mathrm{Fr}(S_0)$, *and*
- $\mathrm{Aut}(S_0)$ *is a 2-group.*

Then there are no reduced fusion systems over $A \times S_0$ for any finite abelian 2-group $A \neq 1$.

We have also shown a few other, much more specialized results of this type.