Endotrivial modules

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Set-up

- p a prime;
- $k = \overline{k}$ a field of characteristic p;
- G a finite group;
- kG-module = finitely generated left kG-module;
- For kG-modules M and N:
  (i) $\mathrm{Hom}_k(M, N) = \{ \varphi : M \to N \mid \varphi \text{ is } k\text{-linear} \}$;
  (ii) $M^* = \mathrm{Hom}_k(M, k)$ with $(g \cdot \varphi)(m) = \varphi(g^{-1} \cdot m)$
  (iii) $M \otimes N = M \otimes_k N$ with $g \cdot (m \otimes n) = (g \cdot m) \otimes (g \cdot n)$

  for all $g \in G$, all $\varphi \in \mathrm{Hom}_k(M, N)$, all $m \in M$, and all $n \in N$.

RECALL :

$$\mathrm{Hom}_k(M, N) \cong M^* \otimes N \quad \text{as } kG\text{-modules.}$$

Endotrivial modules

Let M be a kG-module. Then M is endotrivial if

$$\mathrm{End}_k\, M \cong k \oplus \ (\mathrm{proj}) \quad \text{as } kG\text{-modules.}$$

Equivalently, M is endotrivial if $M^* \otimes M \cong k \oplus \ (\mathrm{proj})$.
Hence, e-t modules are 'invertible' in $\mathbf{stmod}(kG)$.
For short, 'endotrivial' = 'e-t'.

- If M is e-t, then $M = M_0 \oplus \ (\mathrm{proj})$, with M_0 indecomposable e-t. We call M_0 the cap of M.
- For M, N e-t, set $M \sim N$ if there are projective modules P and Q such that $M \oplus P \cong N \oplus Q$. That is,

$$M \sim N \iff M \cong N \quad \text{in} \quad \mathbf{stmod}(kG)$$
$$\iff M^* \otimes N \cong k \oplus \ (\mathrm{proj}) \quad \text{in} \quad \mathbf{mod}(kG) \ .$$

The group of endotrivial modules

The set

$$T(G) = \{ \, [M] \mid M \text{ is an e-t } kG\text{-module} \, \}$$

of equivalence classes of e-t modules is the

group of endotrivial modules of G.

- The composition law is : $[M] + [N] = [M \otimes N]$.
- Hence $0 = [k]$ and $-[M] = [M^*]$.

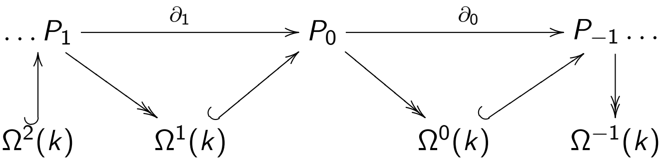
Examples

(i) Trivial: k is an indecomposable e-t kG-module.

(ii) Constructive: Let $0 \to N \to P \to M \to 0$ be a short exact sequence of kG-modules, with P projective. Then,

$$M \quad \text{is e-t} \quad \iff \quad N \quad \text{is e-t.}$$

(iii) 'Ω': If $P \xrightarrow{\partial} k$ is a projective cover of k, then $\Omega(k) = \mathrm{Ker}(\partial)$ is e-t and indecomposable. Inductively, with $\Omega(k) = \Omega^1(k)$ and $k = \Omega^0(k)$, we get an indecomposable e-t module $\Omega^n(k)$ for each $n \in \mathbb{Z}$, called the *n-th syzygy of k* :



Note that, $n[\Omega(k)] = [\Omega^n(k)]$ in $T(G)$, for all $n \in \mathbb{Z}$.

Detection

Set $\mathcal{E}(G)$ for the poset of nontrivial elementary abelian p-subgroups of G, and $\mathcal{E}_{\geq 2}(G) = \{ E \in \mathcal{E}(G) \mid |E| \geq p^2 \}$.

(i) For any $E \in \mathcal{E}(G)$,

$$T(E) = \langle \Omega(k) \rangle \cong \begin{cases} 0 & \text{if } |E| = p = 2; \\ \mathbb{Z}/2 & \text{if } |E| = p > 2; \\ \mathbb{Z} & \text{if } E \text{ is not cyclic.} \end{cases}$$

(ii) A kG-module M is e-t if and only if $\mathrm{Res}^G_E M$ is e-t for all $E \in \mathcal{E}(G)$.

(iii) (Puig) The map

$$\prod_{E \in \mathcal{E}(G)} \mathrm{Res}^G_E \; : \; T(G) \to \prod_{E \in \mathcal{E}(G)} T(E)$$

has finite kernel.

Group structure

$T(G)$ is a finitely generated abelian group. Hence,

$$T(G) = TT(G) \oplus TF(G) \quad \text{with}$$

- $TT(G)$ the torsion subgroup of $T(G)$ and $|TT(G)| < \infty$.
- $TF(G)$ a torsion-free subgroup direct sum complement of $TT(G)$ in $T(G)$.

$TT(G)$ vs $TF(G)$

Note that

(i) $TT(G)$ is unique, but $TF(G)$ is not unique!

(ii) $[M] \in TT(G) \;\Leftrightarrow\; M^{\otimes n} = k \oplus$ (proj) for some $n \in \mathbb{N}$.

About '$TF(G)$'

Fix G and let $\mathcal{E}_{\geq 2}(G)$ the poset of non-cyclic elementary abelian p-subgroups of G with '$\leq$' = '$\subseteq$'. Fact:

$$\mathcal{E} = \mathcal{B} \cup \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_l$$

where $\mathcal{C}_i = \{E_i\}$ for some maximal element E_i of $\mathcal{E}_{\geq 2}(G)$ with $|E_i| = p^2$, and where $E \in \mathcal{B}$ for all $E \in \mathcal{E}_{\geq 2}(G)$ with $|E| \geq p^3$ (possibly none). Also, if $\mathcal{E}_{\geq 2}(G) \neq \emptyset$, then $\exists$ $E \in \mathcal{B}$ with $E \trianglelefteq G$ and $|E| = p^2$.

G acts on $\mathcal{E}_{\geq 2}(G)$ by conjugation; the orbits are the components of $\mathcal{E}_{\geq 2}(G)$.

Theorem (Carlson-Thévenaz, Carlson-M-Nakano)

The rank of $TF(G)$ as free $\mathbb{Z}$-module is the number of components of $\mathcal{E}_{\geq 2}(G)$. Moreover, if $TF(G) \cong \mathbb{Z}$, we may choose $TF(G) = \langle [\Omega(k)] \rangle$.

About '$TT(G)$'

Theorem (Carlson-Thévenaz)

Suppose that G is a p-group. Then $TT(G) = 0$ unless G is cyclic of order ≥ 3, generalised quaternion, or semi-dihedral. More precisely:

- *if G is cyclic of order ≥ 3, then* $TT(G) = T(G) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}/2$;
- *if G is generalised quaternion, then* $TT(G) = T(G) = \langle [\Omega(k)], [M] \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$, *where M is self-dual and* $\dim M \equiv 1 + \frac{|P|}{2} \pmod{|P|}$ *(assuming k contains a primitive cube root of 1);*
- *if G is semi-dihedral, then $TT(G) = \langle [L] \rangle \cong \mathbb{Z}/2$, where L is self-dual and* $\dim L \equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

Dade's unpublished...

An unpublished result by Dade yields : "Suppose that G is a finite group with a normal Sylow p-subgroup P. Then, any G-stable e-t kP-module extends to G."
That is, for any e-t kP-module M such that $M \cong {}^{g}M$ for all $g \in G$, there is a kG-module $\widetilde{M}$ such that $\mathrm{Res}^G_P \widetilde{M} \cong M$. Note that $\widetilde{M}$ is e-t over kG.

Consequences

Suppose that G has a normal Sylow p-subgroup P. Then:

- Res^G_P induces $TF(G) \cong TF(P)^{G/P}$, where $[M] \in T(P)^{G/P}$ means $M \cong {}^{g}M$ for all $g \in G$ and all $M \in [M]$.
- $\ker\left(\mathrm{Res}^G_P : T(G) \to T(P)\right) = \langle\, [M] \mid \dim(M) = 1 \,\rangle$.

...and the Green correspondence

Let G be a finite group, P a Sylow p-subgroup and $N = N_G(P)$. Without loss, all the modules below are e-t and indecomposable (i.e. equal to their cap).

$$\begin{array}{ccccl}
G & \leftrightarrow & M_G & & (\widetilde{\mathrm{Res}_N^G} \text{ is induced by } \mathrm{Res}_N^G) \\
\Big| & & \Big\downarrow \widetilde{\mathrm{Res}_N^G} & & \\
N & \leftrightarrow & M_N & & \\
\Big| & & \Big\downarrow \mathrm{Res}_P^N & & \\
P & \leftrightarrow & M_P & & (\text{i.e. } M_P = \mathrm{Res}_P^N M_N)
\end{array}$$

More precisely, M_N and M_G are in Green correspondence, i.e.

$$M_N \mid \mathrm{Res}_N^G M_G \quad \text{and} \quad M_G \mid \mathrm{Ind}_N^G M_N.$$

In particular, the map

$$\mathrm{Res}_N^G \ : \ T(G) \longrightarrow T(N) \quad \text{is injective.}$$

Assembling pieces of the TT-ji$G$saw

First, remark about the entire group $T(G)$ that since $T(N)$ is "known", it "suffices" to determine which indecomposable e-t kN-modules have an e-t kG-Green correspondent. However, this is TOUGH! and so far, there is no answer...
Set

$$\begin{aligned} K(G) &= \ker\left(\operatorname{Res}^G_P : T(G) \to T(P)\right) \\ &= \langle\ [M] \in T(G) \mid \operatorname{Res}^G_P M = k \oplus \ (\text{proj})\ \ \rangle \end{aligned}$$

for the subgroup of $T(G)$ spanned by the classes of all trivial source e-t kG-modules. Note that $K(G) \leq TT(G)$.
Now, except for a few cases $TT(P) = 0$,
and if so, then $TT(G) = K(G)$. But what is $K(G)$ in general?
And what if $TT(P) \neq 0$?

When $TT(P) \neq 0$

Let G be a finite group and P a Sylow p-subgroup of G. Suppose $TT(P) \neq 0$ and $k = \overline{k}$.
There are 3 cases to consider :

- $TT(P) = \langle [\Omega(k)] \rangle \cong \mathbb{Z}/2$ if P is cyclic of order ≥ 3;
- $TT(P) = \langle [\Omega(k)], [M] \rangle \cong \mathbb{Z}/4 \oplus \mathbb{Z}/2$ if $p = 2$ and P is generalised quaternion;
- $TT(P) = \langle [L] \rangle \cong \mathbb{Z}/2$ if $p = 2$ and P is semi-dihedral.

In particular, $\dim M$, $\dim L$ $\equiv 1 + \frac{|P|}{2} \pmod{|P|}$.

Definition

Suppose that $p = 2$. We call exotic an indecomposable e-t kG-module M with

$$\dim M \equiv 1 + \frac{|P|}{2} \pmod{|P|}.$$

Why 'exotic'?

Suppose M is an e-t kG-module (we may assume M indecomposable). From

$$M^* \otimes M \cong k \oplus \text{ (proj)} \quad \text{we get} \quad (\dim M)^2 = 1 + \dim \text{ (proj)}$$

Since $\dim \text{ (proj)} \equiv 0 \pmod{|P|}$ we get

$$\dim M \equiv \begin{cases} \pm 1 \pmod{|P|} & \text{if } p \text{ is odd;} \\ \pm 1 \pmod{\frac{|P|}{2}} & \text{if } p = 2. \end{cases}$$

This accounts for the existence of exotic e-t modules in characteristic 2.
Hence, torsion exotic e-t modules can occur for finite groups with semi-dihedral or generalised quaternion Sylow 2-subgroup.

Do they?

The Theorem (Carlson-M-Thévenaz, 2010)

Suppose that $p = 2$ and $k = \overline{k}$. Let G be a finite group with semi-dihedral or generalised quaternion Sylow 2-subgroup. Write $K(G) = \ker\big(\operatorname{Res}^G_P : T(G) \to T(P)\big)$ for the subgroup of $T(G)$ spanned by the classes of all trivial source e-t kG-modules, as before. There is a split short exact sequence

$$0 \longrightarrow K(G) \longrightarrow T(G) \xrightarrow{\operatorname{Res}^G_P} T(P) \longrightarrow 0$$

Moreover, if P is generalised quaternion and $H = C_G(Z(P))$, then

$$K(H) = \langle [M] \mid \dim M = 1 \rangle \quad \text{and}$$

$$K(G) = \langle [\operatorname{Ind}^G_H M] \mid [M] \in K(H) \rangle.$$

Semi-dihedral Sylow 2-subgroup

Let G be a finite group with semi-dihedral Sylow 2-subgroup.
K. Erdmann's investigation of algebras of semi-dihedral type shows that the stable Auslander-Reiten quiver of kG has a component $\mathbb{Z}D_\infty$. Hence, the *heart* $H_k = \mathrm{Rad}(R_k)/\,\mathrm{Soc}(R_k)$ of the projective cover R_k of k appears in an almost split sequence

$$0 \longrightarrow V \longrightarrow H_k \oplus \ (\mathrm{proj})\ \longrightarrow U \longrightarrow 0$$

where $\quad V^* \cong U \not\cong \Omega^{-1}(k)$.

Theorem

The modules U and V are e-t. Moreover, $\Omega(U)$ is exotic and self-dual.

In addition, a thorough analysis gives that $\ (\mathrm{proj})\ = 0$, and that U and V are uniserial.

Generalised quaternion Sylow 2-subgroup

Let G be a finite group with generalised quaternion Sylow 2-subgroup P with centre $Z = \langle z \rangle \cong C_2$.
Suppose $Z \leq Z(G)$, and set $\overline{G} = G/Z$ and $\overline{P} = P/Z$.
Well-known : $\Omega^4(k) = k$ and $\Omega^2(k)$ is self-dual.

$$\text{Let} \quad M = \Omega^2(k) \quad , \quad M_0 = \{ \, m \in M \mid (z-1)m = 0 \, \}$$

$$\text{and} \quad (z-1)M = \{ \, (z-1)m \mid m \in M \, \}.$$

$$\text{Filtration :} \quad 0 \subset (z-1)M \underbrace{\subset}_{\cong k} M_0 \underbrace{\subset}_{\cong (z-1)M} M.$$

Moreover, $(z-1)M$ and M_0 are $k\overline{G}$-modules.

Deconstruction – Re-construction

As $k\overline{G}$-module, $(z-1)M = L_1 \oplus L_2$ with L_1, L_2 indecomposable and

$$\dim L_1 = \dim L_2 \equiv \frac{|P|}{4} \pmod{\frac{|P|}{2}}.$$

Theorem (CMT)

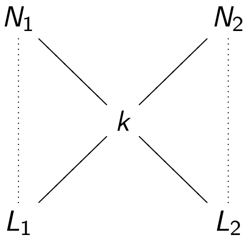
For $i = 1, 2$, there is an exotic e-t kG-module W_i such that

$$(z-1)W_i \cong L_i \quad \textit{as } k\overline{G}\textit{-modules and}$$

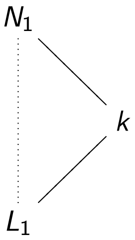
$$\dim W_i \equiv 1 + \frac{|P|}{2} \pmod{|P|}.$$

Visualisation

The diagram for $\Omega^2(k)$ has the form



and so W_1 :



where $N_1 \cong L_1$ is uniserial. Likewise for W_2. (The dotted edges mean that there may or may not be a nontrivial extension between the modules.)

Uniserial e-t modules

Well-known

A block algebra of quaternion type can have $1, 2$ or 3 simple modules.

Using K. Erdmann's description of algebras of quaternion type, a direct inspection of the modules in the case of a block which can occur as principal 2-block of a group algebra (with generalised quaternion Sylow 2-subgroup) gives :

Proposition (CMT)

With the same notation as above, either W_i is uniserial, or $\Omega(W_i)$ is uniserial, or both.

The case $SL_2(q)$, for q odd and $p = 2$

The groups $SL_2(q)$ for q odd have a generalised quaternion Sylow 2-subgroup P. Moreover, there are 3 non-isomorphic simple modules in the principal 2-block: the trivial module k and two non-isomorphic modules of dimension $\frac{q-1}{2}$. Carrying on with the above analysis, we observe the following fact:

Proposition (CMT)

There exist two non-isomorphic uniserial e-t modules of dimension q and composition length 3. More precisely,

- *if $q \equiv 1 \pmod 4$, then these modules are exotic and self-dual;*
- *if $q \equiv 3 \pmod 4$, then their syzygies are exotic, not self-dual, and have dimension $1 + \frac{(q-1)|P|}{8}$.*

What's next?

- About $TT(G)$, we need to determine the subgroup $K(G)$ of $T(G)$ spanned by the classes of the indecomposable trivial source e-t modules.

Puzzling example: the symmetric groups

$$TT(S_n) \cong \begin{cases} 0 & \text{if } n < p; \\ \mathbb{Z}/2(p-1) & \text{if } n = p, p+1; \\ \mathbb{Z}/2(p-1) \oplus \mathbb{Z}/2 & \text{if } p+1 < n < 2p; \\ (\mathbb{Z}/2)^2 & \text{if } 2p \leq n < 3p; \\ \mathbb{Z}/2 & \text{if } 3p \leq n. \end{cases}$$

For $2p \leq n < 3p$, one of the summands $\mathbb{Z}/2$ is given by the class of the Young module labeled by the partition $(n-p, p)$ of n, and which happens to be e-t. The other summand $\mathbb{Z}/2$ is given by the class of the sign representation.

What's next?

- About $TF(G)$, when $TF(G) \ncong \mathbb{Z}$, we need to find e-t modules $M_1, \ldots, M_n$ such that we can choose $TF(G) = \langle [M_i] \mid 1 \leq i \leq n \rangle$. Note that we can always take $M_1 = \Omega(k)$.

Symmetric groups cont'd

Let $G = S_n$ with $p^2 \leq n < p^2 + p$. Then $TF(G) \cong \mathbb{Z}^2$. We can take $M_1 = \Omega(k)$, but we are still missing a suitable M_2...