Modular Representations of Profinite Groups

John MacQuarrie

University of Bristol

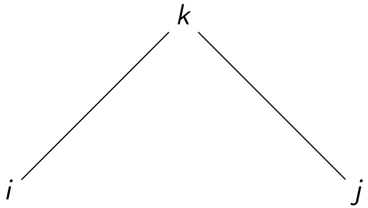
July 5, 2010

Outline

- This is all joint work with Peter Symonds.

Directed Sets

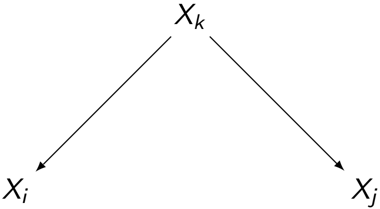
A directed set is a partially ordered set I such that for any $i, j \in I$ there is some $k \in I$ with $k \geq i, j$.



Inverse Systems

An inverse system indexed by I is a set $\{X_i : i \in I\}$ of topological groups (/rings/modules) together with continuous homomorphisms $X_k \to X_i$ whenever $i \leq k$.

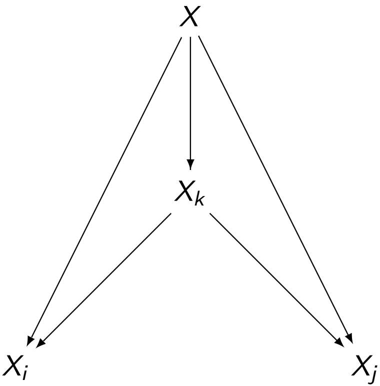
We demand that the map $X_i \to X_i$ is the identity, and that all triangles commute.



Inverse Limits

The inverse limit $X = \varprojlim_{i \in I} X_i$ of this system is a topological group (/ring/module) together with a continuous homomorphism $X \to X_i$ for each $i \in I$.

We demand that any new triangles commute, and that X satisfies an 'obvious' universal property.



Profinite Things

Definition

A profinite group (ring, module) is the inverse limit of an inverse system of finite groups (rings, modules) and group (ring, module) homomorphisms.

Proposition

Let G be a topological group (ring, module). The following are equivalent:

- *G is profinite.*
- *G is compact and the intersection of the open normal subgroups (ideals, submodules) is the identity.*

Idea:

- For any topological group the set $\{G/N \mid N \lhd_O G\}$ forms an inverse system.
- G compact $\implies$ each G/N is finite and $\varprojlim_{N \lhd_O G} G/N = G/\bigcap_N N$
- So if $\bigcap_N N = 1_G$, then $G = \varprojlim G/N$.

Example

The p-adic integers $\mathbb{Z}_p$

- actually form a pro-p group (it's torsion-free, but still very p).
- In fact, since each $\mathbb{Z}/p^n\mathbb{Z}$ is a ring and the maps are ring homs, the p-adic integers form a profinite ring.

$$\begin{array}{c} \vdots \\ \downarrow \\ \mathbb{Z}/p^2\mathbb{Z} \\ \downarrow \\ \mathbb{Z}/p\mathbb{Z} \end{array}$$

Another Profinite Ring

While we're on rings, here's another completely random example:

Definition

k – finite field, G – profinite group. The completed group algebra of G is the profinite k-algebra

$$k[[G]] = \varprojlim_{N \lhd_O G} k[G/N].$$

- The question is: What (finitely generated, indecomposable) profinite $k[[G]]$-modules can we get?

The answer is...

... far too many. But as with finite groups, we can at least organise them:

Definition

$H \leq_C G$. A profinite $k[[G]]$-module U is relatively H-projective if given a diagram in $k[[G]]$-**Mod**

$$\begin{array}{ccc} & & U \\ & & \downarrow \\ V & \twoheadrightarrow & W \end{array}$$

such that there's a continuous $k[[H]]$-module homomorphism $U \to V$ completing the triangle, then there's a continuous $k[[G]]$-module homomorphism $U \to V$ completing the triangle.

A first theorem

If H has infinite index in G, this definition can be hard to work with directly. The following will help:

Theorem (JM)

If U is a finitely generated profinite $k[[G]]$-module, then the following are equivalent:

1. *U is relatively H-projective.*
2. *U is relatively HN-projective for every $N \lhd_O G$.*

□

Vertices

As with finite groups, we can talk about the *minimal* H such that U is H-projective:

Definition

U – a finitely generated indecomposable profinite $k[[G]]$-module. A vertex Q of U is a closed subgroup of G such that

- U is Q-projective
- U is not R-projective for any proper closed subgroup R of Q.

A theorem...

The things that must be true, are true:

Theorem (JM)

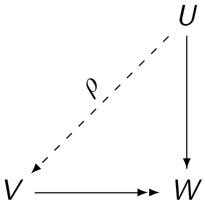
U – a finitely generated indecomposable profinite $k[[G]]$-module, where k has characteristic p:

1. *A vertex of U exists.*
2. *The vertices of U are conjugate in G.*
3. *Any vertex of U is a pro-p group.*

… and part of a proof

Part 3 (Vertices are pro-p): H a p-Sylow subgroup of G (H pro-p with index coprime to p).

Reminder: G finite. We're given



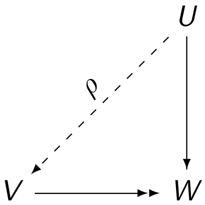
and we define $\rho' : U \to V$ by

$$u \mapsto \frac{1}{|G : H|} \sum_{s \in [G/H]} s \rho s^{-1}(u)$$

… and part of a proof

Part 3 (Vertices are pro-p): H a p-Sylow subgroup of G (H pro-p with index coprime to p).

But if G is PROfinite: We're given



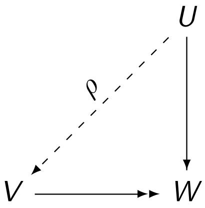
and we're screwed:

$$u \mapsto \frac{1}{|G:H|} \sum_{s \in [G/H]} s\rho s^{-1}(u)$$

… and part of a proof

Part 3 (Vertices are pro-p): H a p-Sylow subgroup of G (H pro-p with index coprime to p).

So instead, let $N \lhd_O G$ be arbitrary, let ρ be a $k[[HN]]$-hom:



and define $\rho' : U \to V$ by

$$u \mapsto \frac{1}{|G : HN|} \sum_{s \in [G/HN]} s\rho s^{-1}(u). \qquad \square$$

More fiddly definitions

Definition

$H \leq_C G$, S a profinite $k[[H]]$-module. The induced $k[[G]]$-module $S\uparrow^G$ is defined as

$$k[[G]]\widehat{\otimes}_{k[[H]]}S.$$

If S is finitely generated, this funny looking tensor product is just a tensor product and $S\uparrow^G$ is exactly what it aught to be.

Definition

U – finitely generated indecomposable $k[[G]]$-module with vertex Q. A source of U is an indecomposable $k[[Q]]$-module S such that

$$U \mid S\uparrow^G .$$

Green Correspondence

Setup:

- G – virtually pro-p group
- Q – closed pro-p subgroup of G
- L – any closed subgroup of G which contains $\mathrm{N}_G(Q)$
- S – f.g. indecomposable $k[[Q]]$-module with vertex Q

Theorem (JM)

There's a canonical bijection between the set of indecomposable profinite $k[[L]]$-modules with vertex Q and source S, and the set of indecomposable profinite $k[[G]]$-modules with vertex Q and source S.

Green Correspondence

A (very) vague idea of the proof:

Lemma

When L is open, there's a word-for-word analogy of the finite version of the Green Correspondence.

Lemma

For general L, given an indecomposable $k[[L]]$-module V with vertex Q and source S, we can find some $N \lhd_O G$ such that $V\uparrow^{LN}$ is indecomposable with vertex Q.

Proof of theorem.

Apply the first lemma to $V\uparrow^{LN}$. This gives the "upwards" map. It's a bijection. □

Green Correspondence

- Question: So why doesn't this theorem look like it should?
- Answer: Because restriction is a badly behaved, ill-mannered functor.
- If U, V are Green correspondents then U is a summand of $V\uparrow^G$, but
- Countertheorem (Symonds) V need NOT be a summand of $U\downarrow_L$.

Weird things like this make profinite modules a mysterious category to explore!

Thanks

References

- "Modular representations of profinite groups"
  (to appear in Pure and Applied Algebra)
- "Green correspondence for virtually pro-p groups"
  (Journal of Algebra)