Finite generation of Hochschild cohomology of Hecke Algebras

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Let A be a finite-dimensional algebra over a field k. The **Hochschild cohomology** of A is the graded k-algebra

$$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$$

where $A^e = A \otimes_k A^{op}$. This algebra is **graded commutative** (Gerstenhaber), thus commutative modulo nilpotent elements. If in addition $HH^*(A)$ is finitely generated as a k-algebra and k is algebraically closed, every finitely generated A-module U gives rise to a **support variety**.

$HH^*(A)$ is known to be finitely generated in the following cases:

- $A = kG$, G a finite group (follows from Evens-Venkov),
- A cocommutative Hopf algebra (Friedlander-Suslin)
- $A = k[x]/(f(x))$, f monic (Holm, Suarez-Alvarez)

There are examples (due to Fei Xu) of algebras A such that even the quotient modulo nilpotent elements in $HH^*(A)$ is not finitely generated.

Theorem

Let $\mathcal{H}$ be the Hecke algebra of a finite Coxeter group W over a field k of characteristic zero with parameter $q \in k^\times$. Suppose that all irreducible components of W are of classical type A, B, D, and suppose in addition that if W has a component of type B or D, the order of q in $k^\times$ is not even. Then the Hochschild cohomology algebra $HH^*(\mathcal{H})$ is finitely generated.

Using results of Dipper-James, Hu, Pallikaros, this is played back to Coxeter groups of type A, in which case one can be more precise:

Theorem

Let $\mathcal{H}$ be the Hecke algebra of a symmetric group S_n over a field k of characteristic zero and parameter $q \in k^\times$ of finite order $\ell \geq 2$. Write $n = m \cdot \ell + a$ for some integers $m \geq 0$ and $0 \leq a \leq \ell - 1$. Then $HH^*(\mathcal{H})$ is finitely generated, with Krull dimension m.

Benson-Erdmann-Mikaelian (2010): calculate

$$\mathrm{Ext}^*_{\mathcal{H}}(k, k)$$

where the hypotheses are as in the first theorem above. The Hochschild cohomology of tame Hecke algebras has been calculated in papers by Erdmann, Schroll, Snashall.

Symmetric Algebras and Adjunction

Definition

An algebra A over a commutative ring k is called **symmetric** if

- $A \cong A^* = \mathrm{Hom}_k(A, k)$ as A-A-bimodules
- A is finitely generated projective as a k-module.

Examples

(1) For G a finite group, the group algebra kG is symmetric; more precisely, the map sending $\mu \in (kG)^*$ to the element $\sum_{x \in G} \mu(x^{-1})x$ in kG is a bimodule isomorphism.

(2) The Hecke algebra $\mathcal{H}$ as above is symmetric.

(3) Algebras of the form $k[x]/(f(x))$, with f monic, are symmetric.

Let A and B be symmetric algebras over a commutative ring k. Let M be an A-B-bimodule such that M is finitely generated projective both as a left A-module and as a right B-module. Then $M^* = \mathrm{Hom}_k(M, k)$ is finitely generated projective as a left B-module and as a right A-module. The functors

$$M \otimes_B - \ : \ \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A)$$

$$M^* \otimes_A - \ : \ \mathrm{Mod}(A) \longrightarrow \mathrm{Mod}(B)$$

are **left and right adjoint** to each other; that is, there are bifunctorial **adjunction isomorphisms**

$$\mathrm{Hom}_A(M \otimes_B V, U) \cong \mathrm{Hom}_B(V, M^* \otimes_A U)$$

$$\mathrm{Hom}_A(U, M \otimes_B V) \cong \mathrm{Hom}_A(M^* \otimes_B U, V)$$

where U is an A-module and B a V-module. Any choice of bimodule isomorphisms $A \cong A^*$ and $B \cong B^*$ determines a choice of adjunction isomorphisms.

Any adjunction isomorphism gives rise to an **adjunction unit** and **counit**, which in the situation above, are represented by bimodule homomorphisms

$$A \to M \otimes_B M^*, \quad M^* \otimes_A M \to B$$

$$B \to M^* \otimes_A M, \quad M \otimes_B M^* \to A$$

Composing the unit of one adjunction with the counit of the other yields bimodule endomorphisms of A and B,

$$A \to M \otimes_B M^* \to A, \quad 1_A \mapsto \pi_M$$

$$B \to M^* \otimes_A M \to B, \quad 1_B \mapsto \pi_{M^*}$$

The images π_M and π_{M^*} of 1_A and 1_B under these endomorphisms are elements in $Z(A)$ and $Z(B)$, respectively. If π_M and π_{M^*} are invertible then all four adjunction maps above are split.

Separably equivalent algebras

Definition

Two symmetric algebras A, B over a commutative ring k are called **separably equivalent** if there is an A-B-bimodule M which is finitely generated projective as a left A-module and as a right B-module, such that, as bimodules,

- A is isomorphic to a direct summand of $M \otimes_B M^*$,
- B is isomorphic to a direct summand of $M^* \otimes_A M$.

Remarks

(1) In the situation of the definition, the adjunction units and counits between A, B and $M \otimes_B M^*$, $M^* \otimes_A M$ split.
(2) If π_M, π_{M^*} are invertible in $Z(A)$ and $Z(B)$, respectively, then A and B are separably equivalent.
(3) If k is an algebraically closed field and A, B are separably equivalent symmetric algebras then A and B have the same representation type (uses a result of Erdmann and Nakano).

Examples

(1) If A, B are Morita equivalent, derived equivalent, stably equivalent of Morita type then A, B are separably equivalent.

(2) A and k are separably equivalent if and only if A is projective as a module over the enveloping algebra $A^e = A \otimes_k A^{op}$.

(3) If k is a field of prime characteristic p and B a block algebra of a finite group algebra kG with a defect group P then B and kP are separably equivalent.

(4) If k is a field of odd characteristic p then the algebra of dual numbers $k[x]/(x^2)$ is not separably equivalent to any finite p-group algebra over k.

Hochschild cohomology

(A finitely generated projective as a k-module):

$$HH^*(A) = \mathrm{Ext}^*_{A^e}(A, A)$$

- $HH^*(A)$ is graded commutative: $\zeta\tau = (-1)^{|\zeta||\tau|}\tau\zeta$;
- for any A^e-module U, the graded k-module $\mathrm{Ext}^*_{A^e}(A, U)$ becomes an $HH^*(A)$-module.

Theorem

Let A, B be separably equivalent symmetric algebras over a field k. The following are equivalent:
(i) $\mathrm{Ext}^*_{A^e}(A, U)$ is Noetherian as an $HH^*(A)$-module, for any finitely generated A^e-module U;
(ii) $\mathrm{Ext}^*_{B^e}(B, V)$ is Noetherian as an $HH^*(B)$-module, for any finitely generated B^e-module V.
If these conditions are satisfied, the Krull dimensions of $HH^*(A)$ and $HH^*(B)$ are equal.

Hecke Algebras

Let (W, S) be a finite **Coxeter group**; that is, W is defined by a generating set S with relations of the form $(st)^{m(s,t)} = 1$ for $s, t \in S$, such that

- $m(s, s) = 1$ for $s \in S$ (that is, the images of the elements of S in W are involutions),
- $m(s, t) = m(t, s) \geq 2$, for $s, t \in S$ such that $s \neq t$.

The *length* $\ell(w)$ of an element $w \in W$ is the smallest integer for which there is an expression of the form $w = s_1 s_2 \cdots s_{\ell(w)}$. By convention, $\ell(1) = 0$. Clearly $\ell(w) = 1$ if and only if $w \in S$, and we have $\ell(w^{-1}) = \ell(w)$ for any $w \in W$.

Example

The Coxeter group of type A_{n-1} is the symmetric group S_n, with generating set $S = \{(1, 2), (2, 3), \ldots, (n-1, n)\}$, where $n \geq 2$.

The **Hecke algebra** $\mathcal{H}(W; q)$ of a finite Coxeter group W over a commutative ring k with parameter $q \in k^\times$ is free as a k-module, with a basis $\{T_w \mid w \in W\}$ indexed by the elements of the group W, with multiplication having T_1 as unit element and satisfying

$$T_w \cdot T_{w'} = T_{ww'} \text{ if } \ell(ww') = \ell(w) + \ell(w')$$

$$(T_s)^2 = qT_1 + (q-1)T_s \text{ for all } s \in S$$

For $q = 1$ we have $\mathcal{H}(W, q) \cong kW$.

If I is a subset of S then the subgroup W_I of W generated by I is a Coxeter group with generating set I and relations from S restricted to I, called a *parabolic subgroup of W*; its Hecke algebra $\mathcal{H}(W_I, q)$ can be identified with a subalgebra of $\mathcal{H}(W, q)$, called a *parabolic subalgebra*.

If S is the disjoint union of pairwise commuting subsets S_1, S_2 then $W \cong W_{S_1} \times W_{S_2}$ and $\mathcal{H}(W, q) \cong \mathcal{H}(W_{S_1}, q) \otimes_k \mathcal{H}(W_{S_2}, q)$.

Let k be a field of characteristic zero, $q \in k^\times$ of finite order $\ell \geq 2$ and let $n \geq 2$. Set $\mathcal{H} = \mathcal{H}(S_n, q)$ and write

$$n = m \cdot \ell + a$$

with $m \geq 0$ and $0 \leq a \leq \ell - 1$. Identify the direct product

$$S_\ell \times S_\ell \times \cdots \times S_\ell$$

of m copies of S_ℓ to a subgroup of S_n; thus

$$\mathcal{H}' = \mathcal{H}(S_\ell \times S_\ell \times \cdots \times S_\ell, q) \cong \mathcal{H}(S_\ell, q)^{\otimes m}$$

is a subalgebra of $\mathcal{H}$, called *ℓ-parabolic subalgebra*.

Proposition

$\mathcal{H}$ and $\mathcal{H}'$ are separably equivalent.

Key ingredients for the proof: a result of J. Du (1992) which shows, by making use of Broué's take on Higman's criterion, that if $M = \mathcal{H}$ viewed as an $\mathcal{H}$-$\mathcal{H}'$-bimodule, then π_M is invertible in $Z(\mathcal{H})$

The proof of the main result is thus played back to the ℓ-parabolic subalgebra $\mathcal{H}' \cong \mathcal{H}(S_\ell, q)^{\otimes m}$. One uses then the fact that $\mathcal{H}(S_\ell, q)$ is a product of a Brauer tree algebra (M. Geck, 1992), and semisimple factors.

Remark

This proves more precisely that if $\mathcal{H}$ is a Hecke algebra as in the first theorem then $\mathrm{Ext}^*_{\mathcal{H}^e}(\mathcal{H}, U)$ is Noetherian as a module over $HH^*(\mathcal{H})$, for any finitely generated $\mathcal{H}^e$-module U.

Questions: What about the Hochschild cohomology in the following situations:

- Hecke algebras of exceptional types,
- fields k of positive characteristic p,
- unequal parameters,
- Ariki-Koike algebras,
- cyclotomic Hecke algebras?

In all cases, the missing ingredient is an analogue of J. Du's result quoted above. For fields of positive characteristic p there is a notion of (ℓ, p)-parabolic subalgebras, which may be a good candidate for generalisations.

Another question:

If $\mathcal{O}$ is a complete discrete valuation ring of characteristic zero with residue field of characteristic $p > 0$, is it true that for two finite p-groups P, Q the algebras $\mathcal{O}P$, $\mathcal{O}Q$ are separably equivalent if and only if $P \cong Q$?

Same question, with P, Q abelian ?

If true this would imply that any two Morita or derived equivalent block algebras of finite groups over $\mathcal{O}$ would have isomorphic (abelian) defect groups.

Bon Anniversaire, Jacques