The Depth of Subgroups and Subrings

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Group Representation Theory
and Related Topics

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This is a report on joint work with

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**Origin:** Operator algebras

**Background:**
A ring extension consists of (associative unitary) rings $B \subseteq A$ such that $1_B = 1_A$.

Main example in this talk:

$$RH \subseteq RG$$

where $H$ is a subgroup of a finite group $G$ and $R \neq 0$ is a commutative ring.
A ring extension $B \subseteq A$ has **depth 1** if

$$A \mid B^n \text{ in } B\text{Mod}_B$$

for some $n \in \mathbb{N}$.

(“$|$” means: “is isomorphic to a direct summand of”)

A ring extension $B \subseteq A$ has **depth 2** if

$$A \otimes_B A \mid A^n \text{ in } A\text{Mod}_B \text{ and } B\text{Mod}_A$$

for some $n \in \mathbb{N}$.

**Remark:** depth 1 $\implies$ depth 2
For categories $\mathcal{C}$, $\mathcal{D}$ and functors $\Phi, \Psi : \mathcal{C} \to \mathcal{D}$, we write $\Phi | \Psi$ if there are natural transformations $\iota : \Phi \to \Psi$, $\pi : \Psi \to \Phi$ such that $\pi \circ \iota = \text{id}_\Phi$.

**Proposition**

A ring extension $B \subseteq A$ has depth 2 $\iff$ there is $n \in \mathbb{N}$ such that

$$\text{Res}_B^A \text{Ind}_B^A \text{Res}_B^A | (\text{Res}_B^A)^n,$$

as functors $A \text{Mod} \to B \text{Mod}$ and $\text{Mod}_A \to \text{Mod}_B$

$\iff$ there is $n \in \mathbb{N}$ such that

$$\text{Ind}_B^A \text{Res}_B^A \text{Ind}_B^A | (\text{Ind}_B^A)^n,$$

as functors $B \text{Mod} \to A \text{Mod}$ and $\text{Mod}_B \to \text{Mod}_A$. 

$R \neq 0$ commutative ring
$H \leq G$ finite groups
$RH \subseteq RG$ group algebras

**Proposition (BOLTJE - K. 2010)**

The ring extension $RH \subseteq RG$ has depth 2 iff $H$ is normal in $G$.

**Remarks:**
(i) This generalizes an earlier result (KADISON - K. 2006) for the case $R = \mathbb{C}$.
(ii) One has a similar result for Hopf algebras.
Corollary (BOLTJE-K.)

- If $G = HC_G(H)$ then the ring extension $RH \subseteq RG$ has depth 1, for any commutative ring $R \neq 0$. The converse, however, does not hold.

- Let $K$ be a field of characteristic 0. Then the ring extension $KH \subseteq KG$ has depth 1 iff $G = HC_G(h)$ for every $h \in H$.

- Let $F$ be a field of characteristic $p > 0$. The ring extension $FH \subseteq FG$ has depth 1 iff $G = HC_G(E)$ for every $p$-hypo-elementary subgroup $E$ of $H$.

Remarks:

(i) Here a group $E$ is called $p$-hypo-elementary if $E$ contains a normal subgroup $N$ of $p$-power order such that $E/N$ is cyclic.

(ii) We do not have a group-theoretic condition characterizing depth 1 extensions $\mathbb{Z}H \subseteq \mathbb{Z}G$ of integral group rings.
Larger depth

\[ B \subseteq A \text{ ring extension} \]

depth 1: \( A \mid B^n \text{ in } B \text{Mod}_B \)

depth 2: \( A \otimes_B A \mid A^n \text{ in } A \text{Mod}_B \text{ and } B \text{Mod}_A \)

depth 3: \( A \otimes_B A \mid A^n \text{ in } B \text{Mod}_B \)

depth 4: \( A \otimes_B A \otimes_B A \mid (A \otimes_B A)^n \text{ in } A \text{Mod}_B \text{ and } B \text{Mod}_A \)

\[ \ldots \]

**Remark:** (i) depth \( n \implies \text{depth } n + 1 \)

So one is usually interested in the **minimal** depth \( d(B, A) \).

(ii) Larger depth can also be expressed via induction and restriction functors, e.g. the depth 3 condition means:

\[ \text{Res}_B^A \text{Ind}_B^A \text{Res}_B^A \text{Ind}_B^A \mid (\text{Res}_B^A \text{Ind}_B^A)^n, \]

for some \( n > 0 \).
$R \neq 0$ commutative ring
$H \leq G$ finite groups

**Theorem (Boltje-Danz-K.)**

The ring extension $RH \subseteq RG$ has finite depth. More precisely,

$$d(RH, RG) \leq 2k - 1 < \infty$$

where $k$ denotes the number of conjugacy classes of subgroups of $H \times H$. 
Bisets

$G, H, K$ finite groups

**Definition.**
A **$G$-$H$-biset** is a (finite) set $X$, together with a left $G$-action and a right $H$-action, such that both actions commute. Every $G$-$H$-biset $X$ can be considered as a left $G \times H$-set via

$$(g, h) \cdot x := g \cdot x \cdot h^{-1} \quad \text{for} \quad g \in G, \ h \in H, \ x \in X,$$

and conversely. For a $G$-$H$-biset $X$ and an $H$-$K$-biset $Y$, one can define a $G$-$K$-biset

$$X \times_H Y := \{[x, y] : x \in X, \ y \in Y\}$$

where $[x, y]$ is the $H$-orbit of $(x, y)$ under the action defined by

$$(x, y) \cdot h := (x \cdot h, h^{-1} \cdot y) \quad \text{for} \quad x \in X, \ h \in H, \ y \in Y.$$
The $G$-$H$-bisets form a category $G\text{Set}_H$; the morphisms are the $G$-$H$-equivariant maps.

Iterating the construction on the previous slide one can define the $G$-$G$-biset

$$\Theta_n := \Theta_n(H, G) := G \times_H \cdots \times_H G \quad (n \text{ factors}),$$

for $n > 0$. In addition, we have the $H$-$H$-biset

$$\Theta_0 := \Theta_0(H, G) := H.$$

Then $RG \otimes_{RH} \cdots \otimes_{RH} RG = R\Theta_n(H, G)$ for every $n$.

Moreover, we have chains of morphisms in $H\text{Set}_G$ and $G\text{Set}_H$:

$$\Theta_1 \hookrightarrow \Theta_2 \hookrightarrow \cdots$$
$H \leq G$ finite groups

**Definition.**
If $n \geq 0$ and $\Theta_{n+1} \hookrightarrow (\Theta_n)^m$ in $\mathcal{HSet}_H$ for some $m > 0$ then $H$ is said to have depth $2n + 1$ in $G$.
If $n > 0$ and $\Theta_{n+1} \hookrightarrow (\Theta_n)^m$ in $\mathcal{HSet}_G$ and $\mathcal{GSet}_H$ for some $m > 0$ then $H$ is said to have depth $2n$ in $G$.

**Remarks.**
(i) Here $X^m$ denotes the disjoint union of $m$ copies of $X$.
(ii) Depth $d$ implies depth $d + 1$. So one is usually interested in the **minimal** depth $d(H, G)$.
Depth of subgroups

$H \leq G$ finite groups
$R \neq 0$ commutative ring

**Theorem (BOLTJE-DANZ-K.)**

Let $k$ be the number of conjugacy classes of subgroups of $H \times H$. Then

$$d(RH, RG) \leq d(H, G) \leq 2k - 1 < \infty.$$ 

In particular, $H$ has always finite depth in $G$.

**Theorem (BOLTJE-DANZ-K.)**

(i) $d(H, G) = 1$ if and only if $G = H\text{C}_G(H)$.
(ii) $d(H, G) \leq 2$ if and only if $H$ is normal in $G$.
(iii) $d(H, G) \leq 3$ if $H$ is a TI-subgroup of $G$, i.e.

$$H \cap gHg^{-1} \in \{H, 1\} \text{ for } g \in G.$$
Depth of subalgebras

$R, S$ non-zero commutative rings  
$H \leq G$ finite groups

**Proposition**

(i) If $R \rightarrow S$ is a unitary ring homomorphism then

$$d(SH, SG) \leq d(RH, RG).$$

(ii) If $R \subseteq S$ is a field extension then $d(SH, SG) = d(RH, RG)$.

One can therefore define $d_0(H, G) := d(KH, KG)$ where $K$ is a field of characteristic 0, and $d_p(H, G) := d(FH, FG)$ where $F$ is a field of characteristic $p > 0$. Then

$$d_0(H, G) \leq d_p(H, G) \leq d(\mathbb{Z}H, \mathbb{Z}G) \leq d(H, G).$$
Examples

Symmetric groups

d\left( S_n, S_{n+1} \right) = 2n - 1 \text{ and } d\left( RS_n, RS_{n+1} \right) = 2n - 1 \text{ for every } n \geq 1 \text{ and every commutative ring } R \neq 0.

Alternating groups

- \quad d\left( A_n, A_{n+1} \right) = 2n - 3 \text{ for } n \geq 2.
- \quad d_0\left( A_n, A_{n+1} \right) = 2n - 2\left\lceil \sqrt{n} \right\rceil + 1 \text{ for } n \geq 2.
- \quad We don’t know (yet) the values of } d_p\left( A_n, A_{n+1} \right) \text{ for } p > 0.
Complex group algebras

$H \leq G$ finite groups
$Irr(G) = \{\chi_1, \ldots, \chi_r\}$
$Irr(H) = \{\psi_1, \ldots, \psi_s\}$

Write

$$\text{Ind}_H^G(\psi_j) \cong \bigoplus_{i=1}^{r} m_{ij} \chi_i \quad (j = 1, \ldots, s).$$

Frobenius reciprocity:

$$\text{Res}_H^G(\chi_i) \cong \bigoplus_{j=1}^{s} m_{ij} \psi_j \quad (i = 1, \ldots, r).$$

Then $M := (m_{ij}) \in \mathbb{N}_0^{r \times s}$ is called the inclusion matrix of the ring extension $\mathbb{C}H \subseteq \mathbb{C}G$. 
• $M$ is the matrix of $\text{Ind}_{H}^{G} : K_{0}(H) \rightarrow K_{0}(G)$
• $M^{\top}$ is the matrix of $\text{Res}_{H}^{G} : K_{0}(G) \rightarrow K_{0}(H)$
• $M^{2} := M^{\top} M$ is the matrix of $\text{Res}_{H}^{G} \text{Ind}_{H}^{G}$
• $M^{3} := MM^{\top} M$ is the matrix of $\text{Ind}_{H}^{G} \text{Res}_{H}^{G} \text{Ind}_{H}^{G}$
• ...

**Proposition**

The ring extension $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $n \geq 2$ \iff $Z(M^{n+1}) = Z(M^{n-1})$

Here $Z(M) = \{(i,j) : m_{ij} = 0\}$. 
Example:

$A = \mathbb{C}S_3$, \quad \text{Irr}(A): 1, 2, 1$

$B = \mathbb{C}S_2$, \quad \text{Irr}(B): 1, 1$

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 2 & 1 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}, \ldots$$

Thus $Z(M^4) = \emptyset = Z(M^2)$, so the minimal depth is 3.
Inclusion graph $\Gamma$ (Bratteli diagram)

Vertices: $\chi_1, \ldots, \chi_r, \psi_1, \ldots, \psi_s$

Edges: $\chi_i \rightarrow \psi_j : \iff m_{ij} \neq 0$

Example: $\mathbb{C}S_2 \subseteq \mathbb{C}S_3$

Proposition

The ring extension $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $2n + 1$ iff any two irreducible characters of $H$ in the same connected component have distance at most $2n$ in $\Gamma$. 
Complex group algebras

$H \leq G$ finite groups
$K := \text{Core}_G(H) := \bigcap_{g \in G} gHg^{-1} \leq G$ core of $H$ in $G$
$\Gamma$ inclusion graph of $\mathbb{C}H \subseteq \mathbb{C}G$

**Proposition**

(i) There is a bijection

$$\{\text{connected components of } \Gamma\} \leftrightarrow \{G\text{-orbits on } \text{Irr}(K)\}.$$

(ii) If $K$ is the intersection of $n$ conjugates of $H$ in $G$ then $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $2n$ (even $2n - 1$ if $K \subseteq Z(G)$).
Complex group algebras

Example:

$G = S_4$, \hspace{2cm} \text{Irr}(G): 1, 3, 2, 3, 1

$H = D_8$ (dihedral group), \hspace{2cm} \text{Irr}(H): 1, 1, 1, 1, 2

$K = V_4$ (Klein four group), \hspace{2cm} \text{Irr}(K): \{1\}, \{1, 1, 1\}

2 connected components, minimal depth 4.
Happy Birthday, Jacques