The Depth of Subgroups and Subrings

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Group Representation Theory and Related Topics

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This is a report on joint work with

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Origin: Operator algebras

Background:
F. M. Goodman, P. de la Harpe, V. F. R. Jones,
Coxeter graphs and towers of algebras,
Springer-Verlag, New York 1989

A **ring extension** consists of (associative unitary) rings $B \subseteq A$ such that $1_B = 1_A$.

Main example in this talk:

$$RH \subseteq RG$$

where H is a subgroup of a finite group G and $R \neq 0$ is a commutative ring.

Depth 1 and 2

A ring extension $B \subseteq A$ has **depth 1** if

$$A \mid B^n \quad \text{in} \quad {}_B\mathbf{Mod}_B$$

for some $n \in \mathbb{N}$.

("$|$" means: "is isomorphic to a direct summand of")

A ring extension $B \subseteq A$ has **depth 2** if

$$A \otimes_B A \mid A^n \quad \text{in} \quad {}_A\mathbf{Mod}_B \quad \text{and} \quad {}_B\mathbf{Mod}_A$$

for some $n \in \mathbb{N}$.

Remark: depth 1 $\Longrightarrow$ depth 2

For categories $\mathcal{C}$, $\mathcal{D}$ and functors $\Phi, \Psi \colon \mathcal{C} \longrightarrow \mathcal{D}$, we write $\Phi \mid \Psi$ if there are natural transformations $\iota \colon \Phi \longrightarrow \Psi$, $\pi : \Psi \longrightarrow \Phi$ such that $\pi \circ \iota = \mathrm{id}_\Phi$.

Proposition

A ring extension $B \subseteq A$ has depth 2
$\iff$ there is $n \in \mathbb{N}$ such that

$$\mathrm{Res}_B^A \mathrm{Ind}_B^A \mathrm{Res}_B^A \mid (\mathrm{Res}_B^A)^n,$$

as functors ${}_A\mathbf{Mod} \longrightarrow {}_B\mathbf{Mod}$ and $\mathbf{Mod}_A \longrightarrow \mathbf{Mod}_B$
$\iff$ there is $n \in \mathbb{N}$ such that

$$\mathrm{Ind}_B^A \mathrm{Res}_B^A \mathrm{Ind}_B^A \mid (\mathrm{Ind}_B^A)^n,$$

as functors ${}_B\mathbf{Mod} \longrightarrow {}_A\mathbf{Mod}$ and $\mathbf{Mod}_B \longrightarrow \mathbf{Mod}_A$.

Group algebras

$R \neq 0$ commutative ring
$H \leq G$ finite groups
$RH \subseteq RG$ group algebras

Proposition (Boltje - K. 2010)

The ring extension $RH \subseteq RG$ has depth 2 iff H is normal in G.

Remarks:
(i) This generalizes an earlier result (Kadison - K. 2006) for the case $R = \mathbb{C}$.
(ii) One has a similar result for Hopf algebras.

Group algebras

Corollary (Boltje-K.)

- If $G = HC_G(H)$ then the ring extension $RH \subseteq RG$ has depth 1, for any commutative ring $R \neq 0$. The converse, however, does not hold.
- Let K be a field of characteristic 0. Then the ring extension $KH \subseteq KG$ has depth 1 iff $G = HC_G(h)$ for every $h \in H$.
- Let F be a field of characteristic $p > 0$. The ring extension $FH \subseteq FG$ has depth 1 iff $G = HC_G(E)$ for every p-hypo-elementary subgroup E of H.

Remarks:

(i) Here a group E is called ***p*-hypo-elementary** if E contains a normal subgroup N of p-power order such that E/N is cyclic.

(ii) We do not have a group-theoretic condition characterizing depth 1 extensions $\mathbb{Z}H \subseteq \mathbb{Z}G$ of integral group rings.

Larger depth

$B \subseteq A$ ring extension

depth 1: $A \mid B^n$ in ${}_B\mathbf{Mod}_B$
depth 2: $A \otimes_B A \mid A^n$ in ${}_A\mathbf{Mod}_B$ and ${}_B\mathbf{Mod}_A$
depth 3: $A \otimes_B A \mid A^n$ in ${}_B\mathbf{Mod}_B$
depth 4: $A \otimes_B A \otimes_B A \mid (A \otimes_B A)^n$ in ${}_A\mathbf{Mod}_B$ and ${}_B\mathbf{Mod}_A$
...

Remark: (i) depth $n \implies$ depth $n+1$
So one is usually interested in the **minimal** depth $d(B, A)$.
(ii) Larger depth can also be expressed via induction and restriction functors, e.g. the depth 3 condition means:

$$\mathrm{Res}_B^A \mathrm{Ind}_B^A \mathrm{Res}_B^A \mathrm{Ind}_B^A \mid (\mathrm{Res}_B^A \mathrm{Ind}_B^A)^n,$$

for some $n > 0$.

Group algebras

$R \neq 0$ commutative ring
$H \leq G$ finite groups

Theorem (Boltje-Danz-K.)

The ring extension $RH \subseteq RG$ has finite depth. More precisely,

$$d(RH, RG) \leq 2k - 1 < \infty$$

where k denotes the number of conjugacy classes of subgroups of $H \times H$.

Bisets

G, H, K finite groups

Definition.
A G-H-**biset** is a (finite) set X, together with a left G-action and a right H-action, such that both actions commute. Every G-H-biset X can be considered as a left $G \times H$-set via

$$(g, h) \cdot x := g \cdot x \cdot h^{-1} \quad \text{for} \quad g \in G, \ h \in H, \ x \in X,$$

and conversely. For a G-H-biset X and an H-K-biset Y, one can define a G-K-biset

$$X \times_H Y := \{[x, y] : x \in X, \ y \in Y\}$$

where $[x, y]$ is the H-orbit of (x, y) under the action defined by $(x, y) \cdot h := (x \cdot h, h^{-1} \cdot y)$ for $x \in X$, $h \in H$, $y \in Y$.

Bisets

The G-H-bisets form a category ${}_G\mathbf{Set}_H$; the morphisms are the G-H-equivariant maps.
Iterating the construction on the previous slide one can define the G-G-biset

$$\Theta_n := \Theta_n(H,G) := G \times_H \cdots \times_H G \quad (n \text{ factors}),$$

for $n > 0$. In addition, we have the H-H-biset

$$\Theta_0 := \Theta_0(H,G) := H.$$

Then $RG \otimes_{RH} \cdots \otimes_{RH} RG = R\Theta_n(H,G)$ for every n.
Moreover, we have chains of morphisms in ${}_H\mathbf{Set}_G$ and ${}_G\mathbf{Set}_H$:

$$\Theta_1 \hookrightarrow \Theta_2 \hookrightarrow \cdots$$

Depth of subgroups

$H \leq G$ finite groups

Definition.
If $n \geq 0$ and $\Theta_{n+1} \hookrightarrow (\Theta_n)^m$ in ${}_H\mathbf{Set}_H$ for some $m > 0$ then H is said to have depth $2n+1$ in G.
If $n > 0$ and $\Theta_{n+1} \hookrightarrow (\Theta_n)^m$ in ${}_H\mathbf{Set}_G$ and ${}_G\mathbf{Set}_H$ for some $m > 0$ then H is said to have depth $2n$ in G.

Remarks.
(i) Here X^m denotes the disjoint union of m copies of X.
(ii) Depth d implies depth $d+1$. So one is usually interested in the **minimal** depth $d(H, G)$.

Depth of subgroups

$H \leq G$ finite groups
$R \neq 0$ commutative ring

Theorem (Boltje-Danz-K.)

Let k be the number of conjugacy classes of subgroups of $H \times H$. Then

$$d(RH, RG) \leq d(H, G) \leq 2k - 1 < \infty.$$

In particular, H has always finite depth in G.

Theorem (Boltje-Danz-K.)

(i) $d(H, G) = 1$ if and only if $G = HC_G(H)$.
(ii) $d(H, G) \leq 2$ if and only if H is normal in G.
(iii) $d(H, G) \leq 3$ if H is a TI-subgroup of G, i.e.

$$H \cap gHg^{-1} \in \{H, 1\} \text{ for } g \in G.$$

Depth of subalgebras

R, S non-zero commutative rings
$H \leq G$ finite groups

Proposition

(i) If $R \longrightarrow S$ is a unitary ring homomorphism then

$$d(SH, SG) \leq d(RH, RG).$$

(ii) If $R \subseteq S$ is a field extension then $d(SH, SG) = d(RH, RG)$.

One can therefore define $d_0(H, G) := d(KH, KG)$ where K is a field of characteristic 0, and $d_p(H, G) := d(FH, FG)$ where F is a field of characteristic $p > 0$. Then

$$d_0(H, G) \leq d_p(H, G) \leq d(\mathbb{Z}H, \mathbb{Z}G) \leq d(H, G).$$

Examples

Symmetric groups

$d(S_n, S_{n+1}) = 2n - 1$ and $d(RS_n, RS_{n+1}) = 2n - 1$ for every $n \geq 1$ and every commutative ring $R \neq 0$.

Alternating groups

- $d(A_n, A_{n+1}) = 2n - 3$ for $n \geq 2$.
- $d_0(A_n, A_{n+1}) = 2n - 2\lceil\sqrt{n}\rceil + 1$ for $n \geq 2$.
- We don't know (yet) the values of $d_p(A_n, A_{n+1})$ for $p > 0$.

$H \leq G$ finite groups
$\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_r\}$
$\mathrm{Irr}(H) = \{\psi_1, \ldots, \psi_s\}$

Write

$$\mathrm{Ind}_H^G(\psi_j) \cong \bigoplus_{i=1}^{r} m_{ij}\chi_i \quad (j = 1, \ldots, s).$$

Frobenius reciprocity:

$$\mathrm{Res}_H^G(\chi_i) \cong \bigoplus_{j=1}^{s} m_{ij}\psi_j \quad (i = 1, \ldots, r).$$

Then $M := (m_{ij}) \in \mathbb{N}_0^{r \times s}$ is called the **inclusion matrix** of the ring extension $\mathbb{C}H \subseteq \mathbb{C}G$.

Complex group algebras

- M is the matrix of $\mathrm{Ind}_H^G \colon \mathrm{K}_0(H) \longrightarrow \mathrm{K}_0(G)$
- $M^\top$ is the matrix of $\mathrm{Res}_H^G \colon \mathrm{K}_0(G) \longrightarrow \mathrm{K}_0(H)$
- $M^2 := M^\top M$ is the matrix of $\mathrm{Res}_H^G \mathrm{Ind}_H^G$
- $M^3 := M M^\top M$ is the matrix of $\mathrm{Ind}_H^G \mathrm{Res}_H^G \mathrm{Ind}_H^G$
- ...

Proposition

The ring extension $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $n \geq 2$
$\iff Z(M^{n+1}) = Z(M^{n-1})$

Here $Z(M) = \{(i,j) : m_{ij} = 0\}$.

Example:

$A = \mathbb{C}S_3,$ $\mathrm{Irr}(A)$: 1, 2, 1

$B = \mathbb{C}S_2,$ $\mathrm{Irr}(B)$: 1, 1

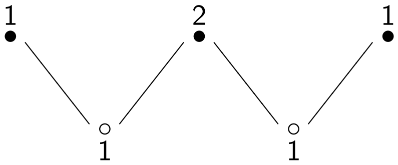
$$M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad M^3 = \begin{pmatrix} 2 & 1 \\ 3 & 3 \\ 1 & 2 \end{pmatrix}, \ldots$$

Thus $Z(M^4) = \emptyset = Z(M^2)$, so the minimal depth is 3.

Inclusion graph Γ (Bratteli diagram)

Vertices: $\chi_1, \ldots, \chi_r, \psi_1, \ldots, \psi_s$
Edges: χ_i —— ψ_j $:\Longleftrightarrow m_{ij} \neq 0$

Example: $\mathbb{C}S_2 \subseteq \mathbb{C}S_3$



Proposition

The ring extension $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $2n+1$ iff any two irreducible characters of H in the same connected component have distance at most $2n$ in Γ.

Complex group algebras

$H \leq G$ finite groups
$K := \mathrm{Core}_G(H) := \bigcap_{g \in G} gHg^{-1} \trianglelefteq G$ core of H in G
Γ inclusion graph of $\mathbb{C}H \subseteq \mathbb{C}G$

Proposition

(i) There is a bijection

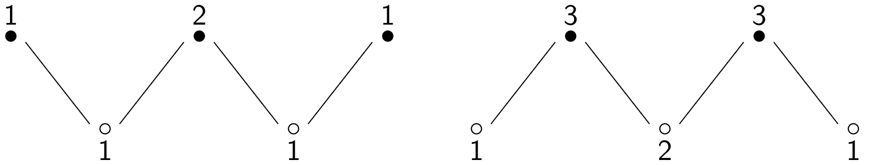
$$\{\text{connected components of } \Gamma\} \longleftrightarrow \{G\text{-orbits on } \mathrm{Irr}(K)\}.$$

(ii) If K is the intersection of n conjugates of H in G then $\mathbb{C}H \subseteq \mathbb{C}G$ has depth $2n$ (even $2n-1$ if $K \subseteq Z(G)$).

Example:

$G = S_4$,
$H = D_8$ (dihedral group),
$K = V_4$ (Klein four group),

$\mathrm{Irr}(G)$: 1, 3, 2, 3, 1
$\mathrm{Irr}(H)$: 1, 1, 1, 1, 2
$\mathrm{Irr}(K)$: $\{1\}$, $\{1, 1, 1\}$



2 connected components, minimal depth 4.

Happy Birthday, Jacques