Broué’s abelian defect group conjecture (1988). Let $p$ be a prime, and let $\mathcal{O}$ be a complete discrete valuation ring whose residue field $k$ is a field of characteristic $p$ such that $k$ is big enough for a finite group $G$. Assume that $A$ is a block algebra of $\mathcal{O}G$ with a defect group $P$ and that $A_N$ is a block algebra of $\mathcal{O}N_G(P)$ which is the Brauer correspondent of $A$, where $N_G(P)$ is the normalizer of $P$ in $G$. Now, if $P$ is abelian, then $A$ and $A_N$ should be derived equivalent, namely,

$$D^b(\text{mod-}A) \cong D^b(\text{mod-}A_N).$$
Theorem. (J. Müller-F. Noeske-S. Koshitani)
Broué's abelian defect group conjecture holds for all primes $p$ and for all block algebras of $O_G$ if $G = \text{Co}_3$, where $O$ is a complete discrete valuation ring whose residue field is of characteristic $p$ and big enough for $G$.

Proposition (F. Noeske). If $A$ is a 2-block of a sporadic finite simple group $G$ such that $A$ is faithful and its defect group $P$ is abelian but is neither cyclic nor Klein four-group, then $A$ is a unique non-principal 2-block of $\text{Co}_3$ and the defect group $P$ is elementary abelian of order 8. To prove Broué's abelian defect group conjecture for $\text{Co}_3$, "relative-projective covers" and "relative-projective Heller operators" do work well.

There are important papers on this subject:


Well-known Lemma.
Let $G$ be a finite group and $H$ a subgroup of $G$, and let $A$ and $B$ be respectively block algebras of $kG$ and $kH$ with a common defect group $P$, where $k$ is a field of characteristic $p > 0$ and big enough for $G$ and $H$. If $M$ is an $(A, B)$-bimodule such that $M$ is a direct summand of $kG \otimes kG_{kH}$ and that $M$ induces a stable equivalence between mod-$A$ and mod-$B$, then, $M$ preserves vertices and sources.

Theorem. (J. Müller-F. Noeske-S. Koshitani)
Let $G = {	ext{Co}}_3$ and let $A$ be a non-principal block algebra of $kG$ with defect group $P = C_2 \times C_2 \times C_2$, where $k$ is a field of characteristic 2 and big enough. Let $H$ be a maximal subgroup of $G$ with $H = R(3) \times S_3 \supseteq N_G(P)$, where $R(3) = 2G_2(3) \cong SL_2(8) \rtimes C_3$ is the smallest Ree group. Let $B$ be a block algebra of $kH$ which is the Brauer
correspondent of $A$. Set $M = f(A)$, where $f$ is the Green correspondence for $(G \times G, \Delta P, G \times H)$. Then, $M$ induces a Morita equivalence between $A$ and $B$, and hence it is a Puig equivalence.


Let $G = \text{Co}_3$ and let $A$ be a non-principal block algebra of $kG$ with defect group $C_2 \times C_2 \times C_2$, where $k$ is a field of characteristic 2 and big enough. Moreover, let $R(q) = ^2G_2(q)$ be a Ree group, where $q = 3^{2n+1}$ for any integer $n \geq 0$, Then, $A$ and $B_0(kR(q))$ are Puig equivalent for all $q = 3, 3^3, 3^5, 3^7, \ldots$, where $B_0(kR(q))$ is the principal block algebra of $kR(q)$.

Thank you very much for your attention and

Jacques,

Happy Birthday to You!