Relative projective cover works for Broué's abelian defect group conjecture

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Joint work with
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Broué's abelian defect group conjecture (1988). Let p be a prime, and let $\mathcal{O}$ be a complete discrete valuation ring whose residue field k is a field of characteristic p such that k is big enough for a finite group G. Assume that A is a block algebra of $\mathcal{O}G$ with a defect group P and that A_N is a block algebra of $\mathcal{O}N_G(P)$ which is the Brauer correspondent of A, where $N_G(P)$ is the normalizer of P in G. Now, if P is abelian, then A and A_N should be derived equivalent, namely,

$$\mathrm{D}^{\mathsf{b}}(\text{mod-}A) \;\simeq\; \mathrm{D}^{\mathsf{b}}(\text{mod-}A_N).$$

Theorem. (J.Müller-F.Noeske-S.Koshitani)
Broué's abelian defect group conjecture holds for all primes p and for all block algebras of $\mathcal{O}G$ if $G = \mathrm{Co}_3$, where $\mathcal{O}$ is a complete discrete valuation ring whose residue field is of characteristic p and big enough for G.

Proposition (F.Noeske). If A is a 2-block of a sporadic finite simple group G such that A is faithful and its defect group P is abelian but is neither cyclic nor Klein four-group, then A is a unique non-principal 2-block of Co_3 and the defect group P is elementary abelian of order 8.
To prove Broué's abelian defect group conjecture for Co_3, "relative-projective covers" and "relative-projective Heller operators" do work well.

There are important papers on th subject:

[1] R. Knörr, Relative projective covers, Proc. Symposium on modular representation of finite groups, Aarhus University, 1978, pp.28–32.

[2] J. Thévenaz, Relative projective covers and almost split sequences, Communications in Algebara **13** (1985), 1533-1554.

Well-known Lemma.
Let G be a finite group and H a subgroup of G, and let A and B be respectively block algebras of kG and kH with a common defect group P, where k is a field of characteristic $p > 0$ and big enough for G and H. If M is an (A, B)-bimodule such that M is a direct summand of ${}_{kG}kG_{kH}$ and that M induces a stable equivalence between mod-A and mod-B, then, M preserves vertices and sources.

Theorem. (J.Müller-F.Noeske-S.Koshitani)
Let $G = \mathsf{Co}_3$ and let A be a non-principal block algebra of kG with defect group $P = C_2 \times C_2 \times C_2$, where k is a field of characteristic 2 and big enough. Let H be a maximal subgroup of G with $H = R(3) \times \mathfrak{S}_3 \geqslant N_G(P)$, where $R(3) = {}^2G_2(3) \cong \mathrm{SL}_2(8) \rtimes C_3$ is the smallest Ree group. Let B be a block algebra of kH which is the Brauer

correspondent of A. Set $M = \mathfrak{f}(A)$, where $\mathfrak{f}$ is the Green correspondence for $(G \times G, \Delta P, G \times H)$. Then, M induces a Morita equivalence between A and B, and hence it is a Puig equivalence.

Corollary. (P.Landrock-G.O.Michler (1980)
$+$ T.Okuyama (1996))
$+$ J.Müller-F.Noeske-S.Koshitani (2010))
Let $G = \mathsf{Co}_3$ and let A be a non-principal block algebra of kG with defect group $C_2 \times C_2 \times C_2$, where k is a field of characateristic 2 and big enough. Moreover, let $R(q) = {}^2G_2(q)$ be a Ree group, where $q = 3^{2n+1}$ for any integer $n \geqslant 0$, Then, A and $B_0(kR(q))$ are Puig equivalent for all $q = 3, 3^3, 3^5, 3^7, ...$, where $B_0(kR(q))$ is the principal block algebra of $kR(q)$.

Thank you very much for your attention and

Jacques,
Happy Birthday to You!