Ordering characters of finite Coxeter groups

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Let W be a finite Coxeter group, with generating set S.
Let $\mathrm{Irr}(W)$ be the set of complex irreducible characters of W.
Write $\mathrm{Irr}(W) = \{\chi^\lambda \mid \lambda \in \Lambda\}$ (for some indexing set Λ).

Example: $W \cong \mathfrak{S}_n$

where $S = \{(1,2), \dots, (n-1, n)\}$; $\Lambda = \{\text{partitions of } n\}$.

Let $L \colon W \to \mathbb{Z}_{\geqslant 0}$ be a weight function in the sense of Lusztig :

- $L(s) = L(t)$ if $s, t \in S$ are conjugate in W ;
- $L(w) = L(s_1) + \cdots + L(s_r)$ where $w = s_1 \cdots s_r$ $(s_i \in S)$ is a reduced expression.

Examples:

- Equal parameter case : $L(s) = 1$, for $s \in S$.
- $W = \langle s, t \rangle$ dihedral group where $(st)^m = 1$ with m even. Fix $a, b \geqslant 0$ and set $L(s) = a$, $L(t) = b$.

Using the generic Iwahori-Hecke algebra, one can define :

- a pre-order relation $\preceq_L$ on $\mathrm{Irr}(W)$ (or on Λ) ;
- an associated equivalence relation :

$$\lambda \sim_L \mu \quad \Leftrightarrow \quad \lambda \preceq_L \mu \ \text{ and } \ \mu \preceq_L \lambda.$$

(These depend on the choice of the weight function L.)

Relevance of these objects :

- Equivalence classes under $\sim_L$: "Lusztig's families".

(Important in the classification of $\mathrm{Irr}(G(q))$ where $G(q)$ finite group of Lie type with Weyl group W ; here $L =$ equal parameter function.)

- Pre-order $\preceq_L$: Generalisation of the well-known dominance order on partitions in the case where $W = \mathfrak{S}_n$.

(Important in the study of "cellular structures" on Iwahori–Hecke algebras in the sense of Graham–Lehrer ; here, consider general L.)

The purpose of this talk is to explain how $\preceq_L$ is determined.

Let $A = \mathbb{Z}[v, v^{-1}]$ and let $\mathcal{H} = \mathcal{H}(W, S, L)$ be the generic Iwahori-Hecke algebra over A with parameters $\{v^{L(s)} \mid s \in S\}$;

$\mathcal{H}$ is an associative algebra, free over A with basis $\{T_w \mid w \in W\}$;

$$\begin{aligned} &T_w = T_{s_1} T_{s_2} \ldots T_{s_p}, \quad \text{if } w = s_1 s_2 \ldots s_p \text{ is reduced} \\ &T_s^2 = (v^{L(s)} - v^{-L(s)}) T_s + T_1 \qquad \text{where } s \in S \end{aligned}$$

Another basis of $\mathcal{H}$, with excellent properties allowing to construct representations of $\mathcal{H}$ and its specialisations : the

Kazhdan-Lusztig basis $\qquad \{C_w \mid w \in W\}$.

Using the Kazhdan-Lusztig basis, we define some relations on W.

- $\leqslant_{\mathcal{L}}$ preorder generated by :
  $x \leqslant_{\mathcal{L}} y$ if C_x appears with non-zero coefficient in $C_s C_y$ for some $s \in S$.

  $x \sim_{\mathcal{L}} y \quad$ if $x \leqslant_{\mathcal{L}} y$ and $y \leqslant_{\mathcal{L}} x$

  equivalence classes called left cells

- $\leqslant_{\mathcal{R}}$, $\sim_{\mathcal{R}}$, right cells are defined similarly (by considering right multiplication by C_s in the defining relation).

  Note that $x \leqslant_{\mathcal{R}} y \Leftrightarrow x^{-1} \leqslant_{\mathcal{L}} y^{-1}$

- $\leqslant_{\mathcal{LR}}$ preorder generated by the union of $\leqslant_{\mathcal{L}}$ and $\leqslant_{\mathcal{R}}$
  $\sim_{\mathcal{LR}}$ and two-sided cells

Any choice of a weight function L gives rise to

- a corresponding Kazhdan-Lusztig basis $\{C_w\}$ of $\mathcal{H}$ and,
- the associated partition of W into cells (left, right, two-sided).

Let $\mathcal{C}$ be a left cell of W. Then the A-modules

$$\mathcal{I}_{\mathcal{C}} = \langle C_y \mid y \leqslant_L w, \text{ for some } w \in \mathcal{C} \rangle_A$$
$$\hat{\mathcal{I}}_{\mathcal{C}} = \langle C_y \mid y \leqslant_L w, \text{ for some } w \in \mathcal{C},\ y \notin \mathcal{C} \rangle_A$$

are left ideals of $\mathcal{H}$. Hence

$[\mathcal{C}]_A := \mathcal{I}_{\mathcal{C}} / \hat{\mathcal{I}}_{\mathcal{C}}$ is a left $\mathcal{H}$-module.

- $[\mathcal{C}]_A$ is free over A, basis $\{b_y \mid y \in \mathcal{C}\}$ (where $b_y = C_y + \hat{\mathcal{I}}_{\mathcal{C}}$).
- Let $\theta\colon A \to \mathbb{C}$ be the specialisation $v \mapsto 1$; then we obtain corresponding "cell modules" $[\mathcal{C}]_1 = \mathbb{C} \otimes_A [\mathcal{C}]_A$ for $\mathbb{C}W$.

Theorem (Lusztig)

For each $\chi \in \mathrm{Irr}(W)$ there exists a left cell $\mathcal{C}$ such that χ appears in the character of $[\mathcal{C}]_1$. All such left cells $\mathcal{C}$ (for χ) are contained in the same two-sided cell.

Hence we can define a map :

$$\mathrm{Irr}(W) \to \text{ two-sided cells}, \quad \chi \mapsto \mathcal{F}_\chi\,.$$

The $\leqslant_{\mathcal{LR}}$ pre-order on W induces a partial order on the set of two-sided cells ; this in turn gives a pre-order relation and an equivalence relation on $\mathrm{Irr}(W)$. Let $\chi, \chi' \in \mathrm{Irr}(W)$, then :

$$\begin{aligned} \chi \leqslant_{\mathcal{LR}} \chi' \quad &\overset{def}{\Longleftrightarrow} \quad \mathcal{F}_\chi \leqslant_{\mathcal{LR}} \mathcal{F}_{\chi'} \\ \chi \sim_{\mathcal{LR}} \chi' \quad &\overset{def}{\Longleftrightarrow} \quad \mathcal{F}_\chi = \mathcal{F}_{\chi'} \end{aligned}$$

EXAMPLE $W = \mathfrak{S}_n$ $L(s_i) = 1$ for all $i = 1, \ldots, n-1$. Then :

- $\mathcal{C}$ left cell $\Longrightarrow$ $[\mathcal{C}]_1$ is irreducible
- $[\mathcal{C}]_1 \cong [\mathcal{C}']_1 \iff \mathcal{C}, \mathcal{C}'$ are in the same two-sided cell
- $\operatorname{Irr}(W) = \{\chi^\lambda \mid \lambda \vdash n\}$.

In this case

$$\begin{array}{rcl} \operatorname{Irr}(W) & \stackrel{1-1}{\longleftrightarrow} & \text{two-sided cells} \\ \chi^\lambda & \longmapsto & \mathcal{F}_\lambda\,, \quad \text{where } w_{\lambda^*} \in \mathcal{F}_\lambda \\ \chi^\lambda \leqslant_{\mathcal{LR}} \chi^\mu & \iff & \lambda \trianglelefteq \mu \end{array}$$

(where λ^* denotes the conjugate partition of λ, w_{λ^*} denotes the longest element of the Young subgroup S_{λ^*}, and $\trianglelefteq$ denotes the dominance order for partitions of n).

An alternative approach to the determination of $\sim_{\mathcal{LR}}$ on $\mathrm{Irr}(W)$. Based on :

- Inductive procedure using parabolic subgroups $\{W_I \mid I \subseteq S\}$;
- **a**-invariant attached to $\chi \in \mathrm{Irr}(W)$;
- Duality $\chi \longmapsto \chi^* = \chi \otimes \varepsilon$ (where ε is the sign character).

Let $K = \mathrm{Quot}(A)$ and $\mathcal{H}_K = K \otimes_A \mathcal{H}$, then

$$\mathrm{Irr}(W) \xrightarrow{1-1} \mathrm{Irr}(\mathcal{H}_K), \quad \chi \longmapsto \chi_v .$$

Let $\tau \colon \mathcal{H}_K \to K$ be the canonical trace defined by $\tau(T_w) = 1$ if $w = 1$, and $\tau(T_w) = 0$ otherwise. Then

$$\tau = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{c_\chi} \chi_v \quad \text{where } 0 \neq c_\chi \in A$$

Now $c_\chi = f_\chi v^{-2\mathbf{a}_\chi} +$ higher powers of v.

Let $I \subseteq S$, W_I the corresponding parabolic subgroup of W : if $E \in \mathrm{Irr}(W)$, $M \in \mathrm{Irr}(W_I)$, write

$$M \rightsquigarrow_L E \quad \overset{def}{\Longleftrightarrow} \quad E \mid \mathrm{Ind}_{W_I}^{W}(M) \text{ and } \mathbf{a}_I(M) = \mathbf{a}(E)$$

(Lusztig) Inductive definition of families :

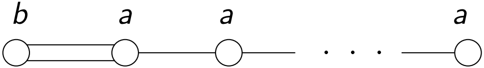
- if $W = \{1\}$, one family $\{1_W\}$;
- assume $W \neq \{1\}$ and families defined for all proper parabolic subgroups of W ;

  E, E' are in the same family of $\mathrm{Irr}(W)$, write $E \sim_L E'$, if there exist $E = E_0, E_1, \ldots, E_m = E' \in \mathrm{Irr}(W)$ such that for each $i = 0, 1, \ldots, m$, there exists $I_i \subsetneq S$ and M_i, M_i' in the same family of $\mathrm{Irr}(W_{I_i})$ such that

$$\begin{aligned} \text{either} \quad & M_i \rightsquigarrow_L E_{i-1} \quad \text{and} \quad M_i' \rightsquigarrow_L E_i \\ \text{or} \quad & M_i \rightsquigarrow_L E_{i-1} \otimes \varepsilon \quad \text{and} \quad M_i' \rightsquigarrow_L E_i \otimes \varepsilon \end{aligned}$$

Examples

- Equal parameter case, $L(s) = 1$, $\forall s \in S$:
  families $\equiv$ equivalence classes of $\mathrm{Irr}(W)$ under $\sim_{\mathcal{LR}}$; (Lusztig)
- Unequal parameters, W of type F_4, $I_2(m)$, with m even :
  families $\equiv$ equivalence classes of $\mathrm{Irr}(W)$ under $\sim_{\mathcal{LR}}$; (Geck)
- Unequal parameters, W of type B_n :



where $a, b > 0$, $\quad b = ra + b'$, $\quad 0 \leqslant b' < a$.
$\mathrm{Irr}(W) = \{E^{(\lambda,\mu)} \mid (\lambda, \mu)$ bipartition of $n\}$.
Lusztig's multisets : $(\lambda, \mu) \longrightarrow Z^N_{a,b}(\lambda, \mu)$, with entries :
$(\lambda_i + N + r - i)a + b', \quad 1 \leqslant i \leqslant N + r,$
$(\mu_j + N - j)a, \quad 1 \leqslant j \leqslant N.$
Then $E^{(\lambda,\mu)}$, $E^{(\lambda',\mu')}$ are in the same family
$\Leftrightarrow Z^N_{a,b}(\lambda, \mu) = Z^N_{a,b}(\lambda', \mu')$. (Lusztig)
The above connection to $\sim_{\mathcal{LR}}$ not yet known.

(Geck) Inductive definition of $\preceq_L$:

- if $W = \{1\}$, we have $1_W \preceq_L 1_W$;
- assume $W \neq \{1\}$ and $\preceq_L$ defined for all proper parabolic subgroups of W ;
  $E, E' \in \mathrm{Irr}(W)$, write $E \preceq_L E'$, if there exist $E = E_0, E_1, \ldots, E_m = E' \in \mathrm{Irr}(W)$ such that for each $i = 0, 1, \ldots, m$, there exists $I_i \subsetneq S$ and $M_i, M_i' \in \mathrm{Irr}(W_{I_i})$ such that $M_i \preceq_L M_i'$ and

$$\begin{array}{rlll} \text{either} & E_{i-1} \mid \mathrm{Ind}_{W_{I_i}}^{W}(M_i) & \text{and} & M_i' \rightsquigarrow_L E_i \\ \text{or} & E_i \otimes \varepsilon \mid \mathrm{Ind}_{W_{I_i}}^{W}(M_i) & \text{and} & M_i' \rightsquigarrow_L E_{i-1} \otimes \varepsilon \end{array}$$

What can we say about $\preceq_L$?

- Equal parameter case : $\preceq_L \quad \equiv \quad \leqslant_{\mathcal{LR}}$ (G.)
- Unequal parameters, W of type F_4 or $I_2(m)$ with m even : $\preceq_L \quad \equiv \quad \leqslant_{\mathcal{LR}}$ (G.)
- Unequal parameters, W of type B_n, $b > (n-1)a$ (the asymptotic case) : $\preceq_L \quad \equiv \quad \leqslant_{\mathcal{LR}}$ (Bonnafé, G., I.)
- Unequal parameters, W of type B_n, remaining choices of $a, b > 0$ (G.-I.) :

$$\begin{aligned} E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \quad &\Longrightarrow \quad E^{(\lambda,\mu)} \leqslant_{\mathcal{LR}} E^{(\lambda',\mu')} \\ E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \quad &\Longrightarrow \quad Z^N_{a,b}(\lambda,\mu) \trianglelefteq Z^N_{a,b}(\lambda',\mu')\,. \end{aligned}$$

Final remarks : W any finite Coxeter group

Let $E, E' \in \mathrm{Irr}(W)$. Then

- $E \sim_L E' \Leftrightarrow E \preceq_L E$ and $E' \preceq_L E$; (G., G.-I.)
- $E \sim_L E' \Rightarrow \mathbf{a}(E) = \mathbf{a}(E')$; (Lusztig)
- $E \preceq_L E' \Rightarrow \mathbf{a}(E') \leqslant \mathbf{a}(E)$. (G., G.-I.)

Next To get the complete description of $\leqslant_{\mathcal{LR}}$ reverse the last two implications on the previous slide ! In other terms obtain the following equivalences :

$$E^{(\lambda,\mu)} \leqslant_{\mathcal{LR}} E^{(\lambda',\mu')} \Leftrightarrow E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \Leftrightarrow Z^N_{a,b}(\lambda,\mu) \trianglelefteq Z^N_{a,b}(\lambda',\mu')\,.$$

These would then give a useful combinatorial tool for proving Lusztig's conjectures P1)-P15) for the remaining cases.