

Minimal Fusion Systems

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Background

- Fusion systems were first studied by Puig, although he called them Frobenius categories rather than fusion systems. His motivation came from block theory.
- The now standard notation and terminology was developed by Broto, Levi and Oliver (2003).

Throughout this talk, p is a prime, S is a finite p -group, and \mathcal{F} is a saturated fusion system on S .

Classification of fusion systems

Motivating question

Can we give a list of all simple saturated fusion systems?

A lot of work approaching this question has been done already by Aschbacher, mainly for the prime 2.

For the classification of fusion systems we can use the experience obtained in the classification of finite simple groups.

p -Local subgroups

Definition

We call a subgroup M of a finite group G a **p -local subgroup** of G if $M = N_G(P)$, for some p -subgroup $P \neq 1$ of G .

A major step in the classification of finite simple groups is the classification of the 2-local subgroups of finite simple groups.

Recall that, by the Theorem of Feit-Thompson, every non-solvable finite group has even order.

p -Fusion systems

Suppose \mathcal{F} is the **p -fusion system** of a finite group G , i.e. $S \in \text{Syl}_p(G)$ and $\mathcal{F} = \mathcal{F}_S(G)$.

Then

$$\text{Mor}_{\mathcal{F}}(Q, Q) \cong N_G(Q)/C_G(Q), \text{ for every } Q \leq S.$$

This means, we mostly see the p -local subgroups of G in \mathcal{F} .

By Alperin's Fusion Theorem, \mathcal{F} is determined by the p -local subgroups of G .

Normalizers

Let $Q \leq S$.

Definition (Puig)

Define $N_{\mathcal{F}}(Q)$ (the **normalizer in \mathcal{F} of Q**) to be the category whose objects are the subgroups of $N_S(Q)$ such that for $A, B \leq N_S(Q)$, $Mor_{N_{\mathcal{F}}(Q)}(A, B)$ is the set of all $\phi \in Mor_{\mathcal{F}}(A, B)$ which extend to an element of $Mor_{\mathcal{F}}(AQ, BQ)$ taking Q to Q .

$N_{\mathcal{F}}(Q)$ is a fusion system on $N_S(Q)$.

Q is normal in \mathcal{F} if and only if $\mathcal{F} = N_{\mathcal{F}}(Q)$.

If \mathcal{F} is the p -fusion system of a finite group G then

$$N_{\mathcal{F}}(Q) = \mathcal{F}_{N_S(Q)}(N_G(Q)).$$

p -Local subsystems

Theorem (Puig)

If Q is fully normalized then $N_{\mathcal{F}}(Q)$ is saturated.

Here Q is called **fully normalized** if $|N_S(Q)| \geq |N_S(P)|$ for all $P \in Q^{\mathcal{F}}$.

Definition

A subsystem of \mathcal{F} is called **p -local** if it is of the form $N_{\mathcal{F}}(Q)$ for some non-trivial fully normalized subgroup Q of S .

Alperin's Fusion Theorem holds accordingly in saturated fusion systems, so a fusion system is determined by its p -local subsystems.

Factor Systems

Recall that a subgroup N of S is called **strongly closed** if, for all $A, B \leq S$ and every morphism $\phi \in \text{Mor}_{\mathcal{F}}(A, B)$, we have $(A \cap N)\phi \leq N$.

Factor systems (Puig)

Given a strongly closed subgroup N of \mathcal{F} we can form a **factor system** \mathcal{F}/N on S/N .

Lemma (Puig)

\mathcal{F}/N is saturated.

Centric subgroups

A subgroup Q of \mathcal{F} is called **centric** if $C_S(P) \leq P$, for every $P \in Q^{\mathcal{F}}$.

Constrained fusion systems

- A fusion system is called **constrained** if it has a normal centric subgroup.
- A **model** for \mathcal{F} is a finite group G such that \mathcal{F} is the p -fusion system of G , and G has a normal p -subgroup containing its centralizer in G .

Theorem (Broto, Castellana, Grodal, Levi, Oliver 2005)

- *If \mathcal{F} is constrained then there exists a model for \mathcal{F} .*
- *A model for \mathcal{F} is always uniquely determined up to isomorphism.*

N-groups

- Minimal fusion systems can be seen as an analog of **N-groups**.
- N-groups were first classified by Thompson. This work set a pattern for the classification of finite simple groups.

N-groups

Thompson classified all finite simple groups with the following property.

(N) Every p -local subgroup is solvable **for every prime p** .

His result was generalized by Gorenstein and Lyons, Janko and Smith to finite simple groups with the following property.

(N2) Every 2-local subgroup is solvable.

Properties of N-groups

- Every minimal non-solvable group is a non-abelian simple N-group.
- Every non-solvable group has a section which is a non-abelian simple N-group.

Minimal Fusion Systems

Definition

The fusion system \mathcal{F} is called **minimal** if $O_p(\mathcal{F}) = 1$ and every p -local subsystem is solvable.

Solvable fusion systems

Definition (Aschbacher)

The fusion system \mathcal{F} is called **solvable** if one of the following equivalent conditions holds.

- 1 Every composition factor has order p .
- 2 $O_p(\mathcal{F}/R) \neq 1$, for every strongly closed subgroup $R \neq S$ of \mathcal{F} .

For example, all p -solvable groups have solvable p -fusion systems.

Solvable fusion systems

Theorem (Aschbacher)

If \mathcal{F} is solvable then \mathcal{F} is constrained.

Corollary

Every minimal fusion system is of characteristic p -type.

Here \mathcal{F} is of **characteristic p -type** if every p -local subsystem is constrained.

Solvable fusion systems

A model for a solvable fusion system can have certain non-abelian finite simple groups as composition factors. (Aschbacher gave a list of them using the classification.)

For example, the p -fusion system of $L_2(q)$ is solvable, for every power q of p .

However, the p -fusion system of $L_n(q)$ is not solvable for $n \geq 3$.

Example $GL_n(q)$

$G = GL_n(q)$ for $q = p^e$ and $n \geq 3$, $S \in \text{Syl}_p(G)$, $\mathcal{F} = \mathcal{F}_S(G)$.
 $S =$ group of lower triangular matrices.

$$Q := \left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & I_{n-1} & & \vdots \\ * & \dots & * & 0 \\ & & & 1 \end{array} \right) \right\} \cong GF(q)^{n-1}$$

$$N_G(Q) = \left\{ \left(\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ * & \dots & * & 0 \\ & & & 1 \end{array} \right) : A \in GL_{n-1}(q) \right\} Z(G)$$

$\implies \mathcal{F}$ is minimal if and only if $n = 3$

Parabolics of groups

Let G be a finite group.

Definition

- A **parabolic subgroup** of G (in short: **parabolic**) is a p -local subgroup of G containing a Sylow p -subgroup of G .
- A **maximal parabolic** of G is a parabolic subgroup of G which is maximal with respect to inclusion among the parabolics of G .

Let S be a Sylow p -subgroup of G . A common case distinction is the following.

- **Case 1:** G has only one maximal parabolic containing S .
- **Case 2:** G has at least two maximal parabolics containing S .

Parabolics in Fusion Systems

Definition

A subsystem of the form $N_{\mathcal{F}}(R)$ for some normal subgroup R of S is called a **parabolic subsystem**.

We call $N_{\mathcal{F}}(R)$ a **full parabolic** if $N_{\mathcal{F}}(S) \leq N_{\mathcal{F}}(R)$.

- **Case 1:** \mathcal{F} has only one full maximal parabolic.
- **Case 2:** \mathcal{F} has at least two full maximal parabolics.

Parabolics in Fusion Systems

Definition

A subsystem of the form $N_{\mathcal{F}}(R)$ for some normal subgroup R of S is called a **parabolic subsystem**.

We call $N_{\mathcal{F}}(R)$ a **full parabolic** if $N_{\mathcal{F}}(S) \leq N_{\mathcal{F}}(R)$.

- **Case 1:** \mathcal{F} has only one full maximal parabolic.
- **Case 2:** \mathcal{F} has at least two full maximal parabolics.

The first result

Theorem

Assume \mathcal{F} is minimal and has only one full maximal parabolic \mathcal{N} . Then there is a centric, fully normalized subgroup Q of \mathcal{F} such that $\text{Mor}_{\mathcal{F}}(Q, Q) \not\leq \mathcal{N}$ and $N_{\mathcal{F}}(Q)$ has the following structure:

For a model G of $N_{\mathcal{F}}(Q)$ and $M := J(G)$ we have

$$M/Q \cong SL_2(q)$$

and one of the following holds.

- (I) Q is elementary abelian, $|Q| \leq q^3$ and $Q/C_Q(M)$ is a natural $SL_2(q)$ -module for M/Q , or*
- (II) $p = 3$, $|Q| = q^5$, $S = N_S(Q)$ and there are two non-central chief factors in Q which are natural $SL_2(q)$ -modules.*

Classification free proof

The proof uses FF-module results from Bundy, Hebbinghaus, Stellmacher, *The Local $C(G, T)$ Theorem* (2005).

It doesn't use the classification of finite simple groups in any form!

$$p = 2$$

If $p = 2$ then one of the following holds

- $|S : N_S(Q)| = 2$, or
- $|Q| = 2^2$ and S is dihedral or semidihedral.

Classification for $p = 2$

Theorem. Suppose $p = 2$. Assume \mathcal{F} is minimal and has a unique full maximal parabolic.

Then \mathcal{F} is the p -fusion system of some finite group G such that one of the following holds:

- $q > 2$ and there is a normal subgroup N of G such that $N \cong SL_3(q)$, $C_G(N) \leq N$ and $|G : N| = 2u$ where u is odd.
- $q = 2^e$ for some odd e , and there is a normal subgroup N of G such that $N \cong Sp_4(q)$, $C_G(N) \leq N$ and $|G : N| = 2u$ where u is odd.
- $q = 4$ and $G \cong J_3$.
- $q = 2$, and $G \cong \text{Aut}(L_3(3))$.

Classification for $p = 2$

- $q = 2$, S is dihedral of order at least 16, and $G \cong L_2(r)$ or $PGL_2(r)$ for some odd prime power r .
- $q = 2$, S is semidihedral of order at least 16, and $G \cong L_2(r^2).2$ for some odd prime power r .
- $q = 2$, S is semidihedral of order 16 and $G \cong L_3(3)$.

Under the \mathcal{K} -group assumption, Aschbacher proved a similar theorem for fusion systems of characteristic 2-type that are not generated by certain full parabolics.

Odd p

Assume p is odd and

- (I) Q is elementary abelian, $|Q| \leq q^3$ and $Q/C_Q(M)$ is a natural $SL_2(q)$ -module for M/Q .

How does S look like?

Odd p

- For every odd p and every $3 \leq n \leq p$, there are examples where $|Q| = q^2$ and S has shape $q^n : q$. These are subsystems of fusion systems constructed by Clelland and Parker.
- For $q = p$, there are examples where $|Q| = q^2$ and S has arbitrarily large order. More precisely, S has shape $(C_{p^n}^i \times C_{p^{n+1}}^j) : C_p$, where $n \geq 1$ is arbitrary and $i + j = p - 1$.
 (These fusion systems arise as subsystems of fusion systems constructed by Broto, Levi and Oliver.)

Thank you!!!