

Homotopical representation theory

Jesper Grodal

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Homotopical generalizations of groups:

$$G \rightsquigarrow BG_{\hat{p}}$$

Group G	p -local (algebraic)	p -local (homotopical)
finite grp	p -fusion \mathcal{F} / p -loc ft \mathcal{L}	$B\mathcal{L} / (B\mathcal{L})_{\hat{p}}$
conn cpt Lie grp		p -compact group BX
Kac-Moody grp		
\vdots		
$Diff(M)$		$B \text{Aut}(M_{\hat{p}})$

Classification of p -compact groups:

p -compact group BX : BX p -complete, $H^*(\Omega BX; \mathbb{F}_p)$ finite.

Theorem (Andersen–G–Møller–Viruel, 200X)

$$\begin{array}{ccc} \{\text{connected } p\text{-compact groups}\} & \xleftrightarrow{1-1} & \{\mathbb{Z}_p\text{-root data}\} \\ BX & \rightsquigarrow & \mathbf{D}_X \end{array}$$

$\mathbf{D}_X = (W_X, L_X, \{\mathbb{Z}_p b_\sigma\}_{\sigma \in \Sigma})$ $L_X = (\mathbb{Z}_p)^r$ integer lattice,
 $W_X \leq \text{Aut}(L_X)$ Weyl group (finite).

$$\mathbf{D} \cong (\mathbf{D}_0 \otimes \mathbb{Z}_p) \times \mathbf{D}_1 \times \cdots \times \mathbf{D}_k$$

\mathbf{D}_0 \mathbb{Z} -root data. $\mathbf{D}_i, i \geq 1$ exotic \mathbb{Z}_p -root data.

$$\{\text{exotic } \mathbb{Z}_p\text{-root data}\} \xleftrightarrow{1-1} \{\text{exotic ft } \mathbb{Q}_p\text{-reflection groups}\} \quad \equiv \quad \text{Navigation icons}$$

Homotopical generalization of representations:

- ▶ Complex linear: $[BG_{\hat{\rho}}, BU(n)_{\hat{\rho}}]$

$$\text{Rep}_{\rho}^{\mathbb{C}}(BG_{\hat{\rho}}) = [BG_{\hat{\rho}}, \coprod_n BU(n)_{\hat{\rho}}] \quad (\text{semiring})$$

- ▶ Real, Quaternionic,...

⋮

- ▶ Modular representation theory: $[BG_{\hat{\rho}}, BGL_n(\mathbb{F}_q)_{\hat{\rho}}]$

Note: $BGL_n(\overline{\mathbb{F}}_{\rho})_{\hat{\rho}}$ contractible. $BGL_n(\overline{\mathbb{F}}_{\ell})_{\hat{\rho}} \simeq BU(n)_{\hat{\rho}}$.

- ▶ Non-linear: $[BG, B\text{Aut}(X_{\hat{\rho}})]$, $X = S^n$ or $X = \text{ft complex}$.

Theorem (Dwyer-Zabrodsky-Notbohm, 80s)

Let $T = (S^1)^r$ torus, H compact Lie grp.

$$\text{Rep}(T, H) \xrightarrow{\cong} [BT, BH]$$

$\check{T} = (\mathbb{Z}/p^\infty)^r$ “discrete approximation”.

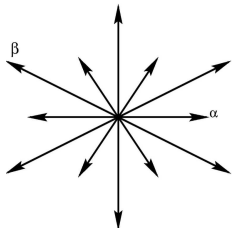
$$\text{Rep}(\check{T}, H) \xrightarrow{\cong} [BT, BH_{\hat{p}}]$$

(similarly with T replaced by “ p -toral”)

$$BC_H(\rho)_{\hat{p}} \xrightarrow{\cong} \text{map}(B\check{T}, BH_{\hat{p}})_\rho$$

Thm(G-Kitchloo '09): Also OK for $H =$ affine Kac–Moody grp.
General Kac–Moody?

Weights:



$$BG \xrightarrow{\phi} BU(n)_{\hat{p}}$$

$$B\check{T} \rightarrow BG \xrightarrow{\phi} BU(n)_{\hat{p}}$$

$$\{\lambda_i : \check{T} \rightarrow S^1\}_{i=1}^n \text{ } W_G\text{-invariant.}$$

$$\{\lambda_i : L_G \otimes \mathbb{Z}_p \rightarrow \mathbb{Z}_p\}_{i=1}^n \text{ } W_G\text{-invariant.}$$

If $p \nmid |W_G|$ W_G -invariance only restriction.

If $p \mid |W_G| \rightsquigarrow$ fusion restrictions.

Given “fusion representation”

$$\rho \in \lim_{P \in \mathcal{S}_\rho(G)}^0 [BP, BU(n)\hat{\rho}]$$

Have spectral sequence:

$$E_2^{i,j} = H_G^{-i}(|\mathcal{S}_\rho(G)|; \pi_j(BC_{U(n)}(-)\hat{\rho})) \Rightarrow \pi_{i+j}(\text{map}(BG, BU(n)\hat{\rho}))$$

Have to deal with “higher fusion”:

Obstructions to existence: on (-1) -line $E^{-(i+1),j}$.

Obstructions to uniqueness: on 0-line $E^{-i,j}$.

E_2 -term can be computed using another spectral sequence involving Steinberg complexes.

The case of $SU(2)$:

$\rho_i = \text{std } (i + 1)\text{-dim cpx rep of } SU(2)$.

$k \star \rho = \text{precompose } \rho \text{ with Adams op } \psi^k \text{ of } BSU(2)\hat{2}, k \in \mathbb{Z}_2^\times$.

Theorem

$\text{Rep}^{\mathbb{C}}(BSU(2)\hat{2})$ has additive generating set

$\rho_0, k \star \rho_1, k \star \rho_2$, and $(k \star \rho_1) \otimes (k' \star \rho_1)$,

$k, k' \in \mathbb{Z}_2^\times$.

Two representations agree if they have the same weights.

Idea of proof:

▶ Look at reps of (ρ -discrete) maximal torus normalizer

▶ require $\rho\left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)$ conj to $\rho\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right)$

(unpleasant) Discoveries:

- ▶ Decomposition not unique for $BSU(2)\hat{2}$ (try ρ_6).
- ▶ Indecomposable reps not well behaved wrt products
No upper bound on dim of $BSU(2)\hat{2} \times BSU(2)\hat{2}$ -reps
- ▶ In general representations not determined by weights
($BSp(2)\hat{2} \times BSp(2)\hat{2}$, Oliver).

Question: Example of fusion representation which does not extend to homotopy representation?

The case of $G = GL_3(\mathbb{F}_2)$:

$D_8 \subseteq G$.

$\text{IrrReps}^{\mathbb{R}}(D_8)$: ρ_{triv} , ρ_a , ρ_b , ρ_{ab} , and ρ_{std} .

$$\lim_{P \in \mathcal{S}_p(G)}^0 [BP, \coprod_n BO(n)\hat{2}] = \mathbb{Z}_+\nu_0 \oplus \mathbb{Z}_+\nu_1 \oplus \mathbb{Z}_+\nu_2$$

$$\nu_0 = \rho_{triv}, \nu_1 = \rho_a + \rho_b + \rho_{std}, \nu_2 = \rho_{ab} + \rho_{std}$$

Theorem

$$\text{Rep}_2^{\mathbb{R}}(BGL_3(\mathbb{F}_2)) =$$

$$(\mathbb{Z}_+\nu_0 + \mathbb{Z}_+\nu_1 + \mathbb{Z}_+\nu_2^+ + \mathbb{Z}_+\nu_2^-) / 2\nu_2^+ = 2\nu_2^-, \nu_1 + \nu_2^+ = \nu_1 + \nu_2^-$$

Taking Grothendieck group:

Def: $R(BG) = Gr([BG, \coprod_n BU(n)])$

Theorem (Jackowski-Oliver '95)

G connected cpt Lie. $R(BG) \cong R(BT)^W \cong R(T)^W \cong R(G)$

Is $Gr([BX, \coprod_n BU(n)_{\hat{\rho}}]) =: R_p(BX) \xrightarrow{\cong?} R_p(BT)^{W_X} = \mathbb{Z}[L_X]^{W_X}$?

Algebraic question: What's the structure of the right-hand side?

Thm (Jeanneret-Osse, 90's): $K^*(BX; \mathbb{Z}_p) \cong K^*(BT; \mathbb{Z}_p)^{W_X}$

How about $[BG, B \text{Aut}(S^n)]$?

Theorem (G-Smith)

P finite p -group

$$\text{Rep}(P, O(n+1))/\Gamma \xrightarrow{\cong} [BP, B \text{Aut}(S^{\widehat{n}})]$$

$\Gamma =$ *galois conjugation*.

G finite group, P p -Sylow.

$$[BG, B \text{Aut}(S^{\widehat{n}})] \hookrightarrow [N_G(P), B \text{Aut}(S^{\widehat{n}})]$$

Algebraic description of image.

Thm: $\text{Rep}(\check{T}, O(n+1))/\Gamma \xrightarrow{\cong} [B\check{T}, B \text{Aut}(S^{\widehat{n}})]$

so weights for reps of compact Lie groups makes sense also in this setting.

$\text{Aut}(X)$, where X simply connected finite complex

Theorem (Dwyer-G, last week)

X simply connected finite complex.

There exists an upper bound on rank r of a monomorphism
 $B(\mathbb{Z}/p)^r \rightarrow B\text{Aut}(X)$

Question: What is the value in terms of X ?

“ p -rank of symmetry”

For S^n it is $n + 1$ by G-Smith.

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