Generic representations of finite groups of Lie type

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Let q be a prime power and consider:

$$\{ \dim \rho \mid \rho \in \mathrm{Irr}_{\mathbb{C}}(\mathrm{SL}_2(q)) \} = ?$$

(whole character table known by Schur, 1907)

q	$\{\dim \rho\}$
2	$1, 1, 2$
3	$1, 1, 1, 2, 2, 2, 3$
4	$1, 3, 3, 4, 5$
5	$1, 2, 2, 3, 3, 4, 4, 5, 6$
7	$1, 3, 3, 4, 4, 6, 6, 6, 7, 8, 8$
8	$1, 7, 7, 7, 7, 8, 9, 9, 9$
$\vdots$	$\vdots$

Observation: For any q,

$\dim \rho = f(q)$ for some

$$f \in \{ 1,\ X,\ X \pm 1,\ \tfrac{1}{2}(X \pm 1) \}$$

"Dimensions are given by polynomials in q"

Consider a series of finite groups of Lie type of a fixed "type":

$$\mathcal{S} = \{G(q) \mid q \text{ any prime power}\}.$$

("Type": Weyl group W + graph automorphism $\gamma \colon W \to W$ + root datum.)

$$\begin{array}{llll}
G(q) = \mathrm{SL}_2(q) & \text{type } A_1: & W \cong \mathfrak{S}_2, & \gamma = \mathrm{id} \\
G(q) = \mathrm{SL}_n(q) & \text{type } A_{n-1}: & W \cong \mathfrak{S}_n, & \gamma = \mathrm{id} \\
G(q) = \mathrm{GL}_n(q) & \text{type } A_{n-1}: & W \cong \mathfrak{S}_n, & \gamma = \mathrm{id} \\
G(q) = \mathrm{SU}_n(q) & \text{type } {}^2A_{n-1}: & W \cong \mathfrak{S}_n, & \gamma^2 = \mathrm{id} \\
\vdots & \vdots & \vdots & \vdots \\
G(q) = E_8(q) & \text{type } E_8: & W = W(E_8), & \gamma = \mathrm{id}
\end{array}$$

Let k be an algebraically closed field. Then describe the set

$$\mathrm{Dim}_k G(q) := \{\dim \rho \mid \rho \in \mathrm{Irr}_k(G(q))\} \text{ as } q \text{ varies.}$$

If $k = \mathbb{C}$, then $\dim \rho$ divides $|G(q)|$, so first look at $|G(q)|$.

Order formula (Chevalley, Solomon, Steinberg).

There exists a polynomial $f \in \mathbb{Z}[X]$ (called the "order polynomial"), which only depends on the "type" of the series $\mathcal{S}$, such that

$$|G(q)| = f(q) \quad \text{for any } q;$$

$$f = X^N \cdot (\text{product of cyclotomic polynomials } \Phi_e(X)).$$

$$\begin{aligned}
|\mathrm{SL}_2(q)| &= q(q^2-1)\\
|\mathrm{SL}_n(q)| &= q^{\frac{1}{2}n(n-1)}(q^2-1)(q^3-1)\cdots(q^n-1)\\
|\mathrm{SU}_n(q)| &= q^{\frac{1}{2}n(n-1)}(q^2-1)(q^3+1)\cdots(q^n-(-1)^n)\\
&\vdots\\
|E_8(q)| &= q^{120}(q^2-1)(q^8-1)(q^{12}-1)(q^{14}-1)\\
&\quad\cdot(q^{18}-1)(q^{20}-1)(q^{24}-1)(q^{30}-1)
\end{aligned}$$

Theorem (Deligne–Lusztig, Lusztig). Let $k = \mathbb{C}$.

There is a **finite** set of polynomials $\mathcal{P} \subseteq \mathbb{Q}[X]$, depending only on the "type" of the series $\mathcal{S}$, such that

$$\mathrm{Dim}_{\mathbb{C}} G(q) \subseteq \{f(q) \mid f \in \mathcal{P}\} \quad \text{for any } q.$$

The polynomials in $\mathcal{P}$ may be called "character degree polynomials"; they are explicitly known for each "type".

Sketch of proof. $G(q) = \mathbf{G}^F$ where $\mathbf{G}$ algebraic group with Weyl group W and $F \colon \mathbf{G} \to \mathbf{G}$ Frobenius map. Using algebraic geometry:

$$(\mathbf{T}, \theta) \quad \rightsquigarrow \quad R_{\mathbf{T},\theta}$$

where:

- $\mathbf{T} \subseteq \mathbf{G}$ any F-stable maximal torus, any $\theta \in \mathrm{Irr}_{\mathbb{C}}(\mathbf{T}^F)$;
- $R_{\mathbf{T},\theta}$ integral linear combination of $\mathrm{Irr}_{\mathbb{C}}(G(q))$.

1. $\pm R_{\mathbf{T},\theta} \in \mathrm{Irr}_{\mathbb{C}}(G(q))$ if θ in "general position"; ("most" irreducible representations of $G(q)$ arise in this way).
2. In general, $\langle R_{\mathbf{T},\theta}, R_{\mathbf{T},\theta}\rangle \leqslant |W|$ and, given $\rho \in \mathrm{Irr}_{\mathbb{C}}(G(q))$, we have $\langle R_{\mathbf{T},\theta}, \rho\rangle \neq 0$ for some $(\mathbf{T}, \theta)$.
3. $\pm \dim R_{\mathbf{T},\theta} = [\mathbf{G}^F : \mathbf{T}^F]_{q'} = f_w(q)$ where $\mathbf{T}$ is of type $w \in W$ and $f_w \in \mathbb{Z}[X]$ only depends on w. Hence, $\{f_w \mid w \in W\} \subseteq \mathcal{P}$.
4. For any $\rho \in \mathrm{Irr}_{\mathbb{C}}(G(q))$:

$$\dim \rho = \sum_{\substack{(\mathbf{T},\theta)/\sim \\ \text{geometrically conjugate}}} \frac{\langle R_{\mathbf{T},\theta}, \rho\rangle}{\langle R_{\mathbf{T},\theta}, R_{T,\theta}\rangle} \dim R_{\mathbf{T},\theta}.$$

$$\rightsquigarrow \quad \mathcal{P} \subseteq \Big\{ \sum_{w\in W} \frac{n_w}{m_w} f_w \mid n_w, m_w \in \mathbb{Z} \text{ such that } |n_w|, |m_w| \leqslant |W| \Big\}.$$

Consider $\mathcal{S} = \{\mathrm{SL}_2(q) \mid q \text{ prime power}\}$. Let ℓ be a prime and $k = \overline{\mathbb{F}}_\ell$.
Assume first that $q = \ell^f$ ("natural" characteristic); then

$$\{1, \ell, \ell^2, \ell^3, \ldots, \ell^f\} \quad \subseteq \quad \mathrm{Dim}_{\overline{\mathbb{F}}_\ell}\mathrm{SL}_2(q)$$

(hence, $\mathrm{Dim}_{\overline{\mathbb{F}}_\ell}\mathrm{SL}_2(q)$ can not be described by polynomials in q !)

$\dim \rho$	Decomposition mod ℓ, $\ell \nmid q$
1	irreducible
q	irreducible if $\ell \nmid q+1$; 1-dim $\oplus$ $(q-1)$-dim if $\ell \mid q+1$.
$q \pm 1$	irreducible
$\frac{1}{2}(q \pm 1)$	irreducible

If $\ell \nmid q$, then $\mathrm{Dim}_{\overline{\mathbb{F}}_\ell}\mathrm{SL}_2(q) \subseteq \mathrm{Dim}_{\mathbb{C}}\mathrm{SL}_2(q)$
(hence, $\mathrm{Dim}_{\overline{\mathbb{F}}_\ell}\mathrm{SL}_2(q)$ can be described by polynomials in q).

Consider $\mathcal{S} = \{\mathrm{GL}_4(q) \mid q \text{ prime power}\}$.

$$\mathrm{Dim}_{\mathbb{C}}\mathrm{GL}_4(q) = \{1,\ q(q^2+q+1),\ q^2(q^2+1),\ q^3(q^2+q+1),\ q^6,\ \ldots\}$$

dim ρ	Decomposition mod ℓ, $\ell \mid q+1$
1	irreducible
$q(q^2+q+1)$	1-dim $\oplus$ (q^3+q^2+q-1)-dim
$q^2(q^2+1)$	(q^3+q^2+q-1)-dim $\oplus$ $(q-1)^2(q^2+q+1)$-dim
$\vdots$	$\vdots$

Note: $f_0 := X^3+X^2+X-1 \in \mathbb{Q}[X]$ is irreducible and not a character degree polynomial for $\mathcal{S}$! However, one can check:

If $\ell \nmid q$, then $\mathrm{Dim}_{\mathbb{F}_\ell}\mathrm{GL}_4(q) \subseteq \mathrm{Dim}_{\mathbb{C}}\mathrm{GL}_4(q) \cup \{f_0(q)\}$.

Source: G.D. James, The decomposition matrices of $\mathrm{GL}_n(q)$ for $n \leqslant 10$.

Proc. London Math. Soc. **60**, 225–265 (1990).

Theorem $(k = \mathbb{C})$ / **Conjecture.**

There is a **finite** set of polynomials $\mathcal{P} \subseteq \mathbb{Q}[X]$, depending only on the "type" of the series $\mathcal{S}$, such that

n$_{\mathbb{C}\overline{\mathbb{F}}_\ell} G(q) \subseteq \{f(q) \mid f \in \mathcal{P}\}$ for any q $\quad q, \ell$ such that $\ell \nmid$

(Note: If $\ell \nmid |G(q)|$, then $\mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G(q))$ can be identified with $\mathrm{Irr}_{\mathbb{C}}(G(q))$ so the conjecture also covers the theorem for $k = \mathbb{C}$.)

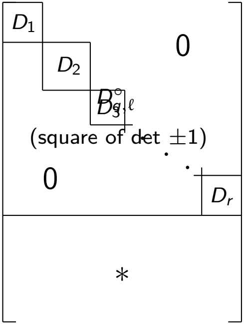
Status of the conjecture:

- Known for $\mathrm{GL}_n(q)$, all q, ℓ (Dipper–James);
- also for $\mathrm{SU}_3(q)$, $\mathrm{Sp}_4(q)$, all q, ℓ (G., White, Okuyama–Waki).
- For other types of groups, with some restrictions on q, ℓ : $G(q)$ of classical type, q mod ℓ has odd order (Gruber–Hiss).

Let $D_{q,\ell}$ be the ℓ-modular decomposition matrix of $G(q)$, $\ell \nmid q$:

(Rows labelled by $\mathrm{Irr}_{\mathbb{C}}(G(q))$, columns labelled by $\mathrm{Irr}_{\overline{\mathbb{F}}_\ell}(G(q))$.)

$D_{q,\ell} =$



G.–Hiss (assuming ℓ is "good"):

$\rightsquigarrow$ $\mathrm{Dim}_{\mathbb{C}}\, G(q) \, + \, D^{\circ}_{q,\ell} \;\Rightarrow\; \mathrm{Dim}_{\overline{\mathbb{F}}_\ell}\, G(q)$

Each D_i is square and $\det(D_i) = \pm 1$

Broué–Michel:

$\rightsquigarrow$ Size of D_i's bounded by $|W|$

(Total size and number of D_i's grow with q)

"Mod ℓ Boundedness" Conjecture.

There exists a constant N, depending only on the "type" of the series $\mathcal{S}$, such that the ℓ-modular decomposition numbers of $G(q) \in \mathcal{S}$, for any q, ℓ such that $\ell \nmid q$, are bounded above by N.

There is a distinguished diagonal block of $D^{\circ}_{q,\ell}$ (where ℓ is "good"):

"Unipotent representations".

Let $D^{\text{unip}}_{q,\ell}$ be the part of $D_{q,\ell}$ with rows and columns labelled by:

$$\text{Unip}_{\mathbb{C}}(G(q)) = \{\rho \in \text{Irr}_{\mathbb{C}}(G(q)) \mid \langle R_{\mathbf{T},1}, \rho\rangle \neq 0 \text{ for some } \mathbf{T}\},$$

$$\text{Unip}_{\overline{\mathbb{F}}_\ell}(G(q)) = \left\{\beta \in \text{Irr}_{\overline{\mathbb{F}}_\ell}(G(q)) \;\middle|\; \begin{array}{l}\beta \text{ is an } \ell\text{-modular constituent} \\ \text{of some } \rho \in \text{Unip}_{\mathbb{C}}(G(q))\end{array}\right\}.$$

Broué–Michel + G.–Hiss $\Rightarrow$ $D^{\text{unip}}_{q,\ell}$ is a square diagonal block of $D^{\circ}_{q,\ell}$.

Bonnafé–Rouquier (2003) $\rightsquigarrow$

The determination of $D^{\circ}_{q,\ell}$ can almost completely be reduced to the determination of $D^{\text{unip}}_{q,\ell}$ for $G(q)$ and for "smaller" groups $H(q)$.

Reduction is complete for $\mathrm{GL}_n(q)$ (Dipper–James) and $\mathrm{GU}_n(q)$.

Lusztig's classification of $\mathrm{Unip}_{\mathbb{C}}(G(q))$.

There exist a finite set $\bar{X}$ and integers $\{m_{\lambda,w} \mid \lambda \in \bar{X}, w \in W\}$, which both only depend on the "type" of the series $\mathcal{S}$, such that for any q:

1. $\exists$ bijection $\bar{X} \stackrel{1-1}{\longrightarrow} \mathrm{Unip}_{\mathbb{C}}(G(q))$, $\quad \lambda \mapsto \rho_\lambda$;
2. $\langle R_{\mathbf{T},1}, \rho_\lambda \rangle = m_{\lambda,w}$ for $\lambda \in \bar{X}$, $w \in W$ with $\mathbf{T}$ of type w.

(The bijection in (1) is almost completely determined by the multiplicities in (2). From (1) and (2) we obtain the degree polynomials for all $\rho \in \mathrm{Unip}_{\mathbb{C}}(G(q))$.)

$$D_{q,\ell}^{\text{unip}} = \begin{bmatrix} * & & * \\ & ? & \\ * & & * \end{bmatrix} \qquad (\text{size } |\bar{X}| \times |\bar{X}|).$$

$\rightsquigarrow$ Principal test case for the "Mod ℓ Boundedness Conjecture".

Examples of matrices $D_{q,\ell}^{\text{unip}}$ (rows labelled by $\text{Unip}_{\mathbb{C}}(G(q)) \leftrightarrow \bar{X}$):

$GL_3(q)$	$\ell \mid q+1$		
(3)	1	.	.
(21)	.	1	.
(111)	1	.	1

$GU_3(q)$	$2 \neq \ell \mid q+1$			
(3)	1		.	.
(21)	.		1	.
(111)	1	2	[2]	1

$\rho_{(21)}$ cuspidal

$GL_4(q)$	$2 \neq \ell \mid q+1$				
(4)	1	.	.	.	.
(31)	1	1	.	.	.
(22)	.	1	1	.	.
(211)	1	1	1	1	.
(1111)	1	.	.	1	1

$Sp_4(q)$	$2,3 \neq \ell \mid q+1$					
1_G	1	.	.	.	.	.
θ_{10}	.	1	.	.	.	.
ε_1	1	.	1	.	.	.
ε_2	1	.	.	1	.	.
refl	.	.	.	.	1	.
St_G	1	2 [2]	1	1	.	1

$\text{Irr}(W) = \{1_W, \varepsilon_1, \varepsilon_2, \varepsilon, \text{refl}\}$,
θ_{10} cuspidal (Srinivasan)

Sources:

GL_3, GL_4 : James;

GU_3 : G., Okuyama–Waki (2002);

Sp_4 : White, Okuyama–Waki.

[2] difficult to determine !

Which ordering of $\mathrm{Unip}_{\mathbb{C}}(G(q))$ (the rows of $D_{q,\ell}^{\mathrm{unip}}$) ?

- First approach: Order by $\dim \rho$. (Works in above examples.)
- Order "generically": Use degrees of degree polynomials.

Dipper–James $\rightsquigarrow$ "proper" partial order for $\mathrm{GL}_n(q)$.

- $\mathrm{Unip}_{\mathbb{C}}(\mathrm{GL}_n(q)) = \{ \rho_\lambda \mid \lambda \text{ partition of } n \}$.
- Dipper–James use dominance order $\trianglelefteq$:

  $\lambda \trianglelefteq \mu \quad \Leftrightarrow \quad \lambda_1 + \lambda_2 + \cdots + \lambda_d \leqslant \mu_1 + \mu_2 + \cdots + \mu_d$ for all $d \geqslant 1$.
- Observation: Consider unipotent classes of $\mathrm{GL}_n(\mathbb{C})$ (matrices with all eigenvalues equal to 1).
  - $\{ O_\lambda \mid \lambda \text{ partition of } n \}$ (Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots$).
  - $O_\lambda \subseteq \overline{O}_\mu$ (Zariski closure) $\Leftrightarrow$ $\lambda \trianglelefteq \mu$.

Character values on unipotent classes:

$GL_3(q)$	(111)	(21)	(3)
(3)	1	1	$1\,1$
(21)	$q(q+1)$	$q\,q$	.
(111)	$q^3\,q^3$	.	.

$GU_3(q)$	(111)	(21)	(3)
(3)	1	1	$1\,1$
(21)	$q(q-1)$	$-q-q$	.
(111)	$q^3\,q^3$	.	.

$Sp_4(q)$	(1111)	(211)	(22)	$(22)'$	(4)
1_G	1	1	1	1	$1\,1$
θ_{10}	$\frac{1}{2}q(q-1)^2$	$-\frac{1}{2}q(q-1)$	.	$q\,q$	.
ε_1	$\frac{1}{2}q(q^2+1)$	$\frac{1}{2}q(q+1)$	.	$q\,q$	.
ε_2	$\frac{1}{2}q(q^2+1)$	$-\frac{1}{2}q(q-1)$	$q\,q$	.	.
refl	$\frac{1}{2}(q+1)^2$	$\frac{1}{2}q(q+1)$	$q\,q$	.	.
St_G	$q^4\,q^4$	.	.	.	.

Idea: Associate to each ρ a unipotent class of $\mathbf{G}$ where $G(q) = \mathbf{G}^F$.

Write $G(q) = \mathbf{G}^F$ and let $O \subseteq \mathbf{G}$ be an F-stable unipotent class.
Let $u \in O^F$ and $A(u) := C_{\mathbf{G}}(u)/C_{\mathbf{G}}(u)^\circ$. Then define:

$$\mathrm{AV}(\rho, O) := \sum_{a \in A(u)} \mathrm{trace}(u_a, \rho) \qquad \text{for } \rho \in \mathrm{Irr}_{\mathbb{C}}(G(q)).$$

(To $a \in A(u)$ one can associate $u_a \in O^F$, well-defined up to conjugacy in $G(q)$.)

Lusztig's (+ G.–Malle's) "Unipotent Support".

Let $\rho \in \mathrm{Irr}_{\mathbb{C}}(G(q))$. Then there is a unique F-stable unipotent class in $\mathbf{G}$, which we shall denote by O_ρ, such that

$$\mathrm{AV}(\rho, O_\rho) \neq 0 \quad \text{and} \quad \mathrm{AV}(\rho, O) = 0 \text{ if } \dim O < \dim O_\rho.$$

(There are examples where $\mathrm{trace}(u, \rho) \neq 0$ with $u \in O$ and $\dim O > \dim O_\rho$.)

$$\mathrm{AV}(\rho, O_\rho) = \pm n_\rho^{-1}\, |A(u)|\, q^{\dim \mathcal{B}_u},$$

where $u \in O_\rho$ and n_ρ is a "small" integer $\leqslant |W|$.

Re-consider the example $G(q) = \mathrm{Sp}_4(q)$:

$\mathrm{Sp}_4(q)$	(1111)	(211)	(22)	(22)′	(4)
1_G	$*$	$*$	$*$	$*$	1
θ_{10}	$*$	$*$	.	q	.
ε_1	$*$	$*$	.	q	.
ε_2	$*$	$*$	q	.	.
refl	$*$	$*$	q	.	.
St_G	q^4	.	.	.	.

((22), (22)′ conjugate in **G**)

$2,3 \neq \ell \mid q+1$						2, 3	
1	.	.	.	.	.	1	.
.	1	.	.	.	.	.	1
1	.	1	.	.	.	1	.
1	.	.	1	.	.	1	.
.	.	.	.	1	.	.	.
1	2	1	1	.	1	1	2

Unipotent support		
1_G	$\mapsto$	$O_{(4)}$
$\theta_{10}, \varepsilon_1, \varepsilon_2$, refl	$\mapsto$	$O_{(22)}$
St_G	$\mapsto$	$O_{(1111)}$

$\rightarrow$

Observation:

Three identity blocks in $D_{q,\ell}^{\mathrm{unip}}$ corresponding to the three possible unipotent supports.

Recall $\mathrm{Unip}_{\mathbb{C}}(G(q)) = \{\rho_\lambda \mid \lambda \in \bar{X}\}$. Write $O_\lambda := O_{\rho_\lambda}$ for $\lambda \in \bar{X}$.

$$\bar{X} \to \{\ F\text{-stable unipotent classes in } \mathbf{G}\ \}, \qquad \lambda \mapsto O_\lambda.$$

"Strong Triangularity" Conjecture (G., Hiss). Assume that ℓ is "good".

There exists a labelling $\mathrm{Unip}_{\mathbb{F}_\ell}(G(q)) = \{\beta_\mu \mid \mu \in \bar{X}\}$ such that:

1. $[\rho_\lambda : \beta_\lambda] = 1$ for all $\lambda \in \bar{X}$;
2. $[\rho_\lambda : \beta_\mu] = 0$ unless $\lambda = \mu$ or $O_\lambda \subsetneq \overline{O}_\mu$ (Zariski closure).

(Note: If such a labelling exists, then it is uniquely determined.)

True for: $\mathrm{GL}_n(q)$ (Dipper–James, q-Schur algebra),
$G_2(q)$ (Hiss, explicit computations),
$\mathrm{GU}_n(q)$ (G., generalised Gelfand–Graev representations),
and some further cases of small rank, $\mathrm{Sp}_4(q)$, ${}^3D_4(q)$, ...

Theorem (G. 2007/09). Assume $G(q)$ "split" (that is, $\gamma = \mathrm{id}$).

The "Strong Triangularity" Conjecture is true as far as decomposition numbers of "principal series" representations are concerned.
(That is, those $\rho \in \mathrm{Unip}_{\mathbb{C}}(G(q))$ and $\beta \in \mathrm{Unip}_{\overline{\mathbb{F}}_\ell}(G(q))$ which admit non-zero vectors fixed by a Borel subgroup $B(q) \subseteq G(q)$.)

- By Dipper, the part of $D_{q,\ell}^{\mathrm{unip}}$ corresponding to principal series representations is determined by the decomposition matrix of the Iwahori–Hecke algebra $H_q(W)$ associated with $G(q)$.
- Then the proof of the theorem relies on:
  1. Existence of "cellular structure" on $H_q(W)$ (in the sense of Graham–Lehrer) $\rightsquigarrow$ natural partial order on principal series representations via Kazhdan–Lusztig cells (2007).
  2. Characterisation in terms of order on unipotent classes (2009).