Generic representations of finite groups of Lie type

Meinolf Geck
Aberdeen University

Lausanne, June 2010
Let $q$ be a prime power and consider:

$$\{ \dim \rho \mid \rho \in \text{Irr}_{\mathbb{C}}(\text{SL}_2(q)) \} = ?$$

(whole character table known by Schur, 1907)

<table>
<thead>
<tr>
<th>$q$</th>
<th>${\dim \rho}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1, 1, 2</td>
</tr>
<tr>
<td>3</td>
<td>1, 1, 1, 2, 2, 2, 3</td>
</tr>
<tr>
<td>4</td>
<td>1, 3, 3, 4, 5</td>
</tr>
<tr>
<td>5</td>
<td>1, 2, 2, 3, 3, 4, 4, 5, 6</td>
</tr>
<tr>
<td>7</td>
<td>1, 3, 3, 4, 4, 6, 6, 6, 7, 8, 8</td>
</tr>
<tr>
<td>8</td>
<td>1, 7, 7, 7, 8, 9, 9, 9</td>
</tr>
</tbody>
</table>

Observation: For any $q$,

$$\dim \rho = f(q)$$

for some

$$f \in \{ 1, X, X \pm 1, \frac{1}{2}(X \pm 1) \}$$

“Dimensions are given by polynomials in $q$”
Consider a series of finite groups of Lie type of a fixed “type”:

\[ S = \{ G(q) \mid q \text{ any prime power} \}. \]

(“Type”: Weyl group \( W \) + graph automorphism \( \gamma: W \to W + \) root datum.)

\[
\begin{align*}
G(q) &= \text{SL}_2(q) & \text{type } A_1 : & W \cong S_2, & \gamma = \text{id} \\
G(q) &= \text{SL}_n(q) & \text{type } A_{n-1} : & W \cong S_n, & \gamma = \text{id} \\
G(q) &= \text{GL}_n(q) & \text{type } A_{n-1} : & W \cong S_n, & \gamma = \text{id} \\
G(q) &= \text{SU}_n(q) & \text{type } ^2A_{n-1} : & W \cong S_n, & \gamma^2 = \text{id} \\
& \vdots & \vdots & \vdots & \vdots \\
G(q) &= E_8(q) & \text{type } E_8 : & W = W(E_8), & \gamma = \text{id}
\end{align*}
\]

Let \( k \) be an algebraically closed field. Then describe the set

\[
\text{Dim}_k G(q) := \{ \dim \rho \mid \rho \in \text{Irr}_k(G(q)) \} \quad \text{as } q \text{ varies.}
\]
If $k = \mathbb{C}$, then $\dim \rho$ divides $|G(q)|$, so first look at $|G(q)|$.

**Order formula** (Chevalley, Solomon, Steinberg).

There exists a polynomial $f \in \mathbb{Z}[X]$ (called the “order polynomial”), which only depends on the “type” of the series $\mathcal{S}$, such that

$$|G(q)| = f(q) \quad \text{for any } q;$$

$$f = X^N \cdot \text{(product of cyclotomic polynomials } \Phi_e(X)).$$

$$
\begin{align*}
|\text{SL}_2(q)| &= q(q^2 - 1) \\
|\text{SL}_n(q)| &= q^{\frac{1}{2}n(n-1)}(q^2 - 1)(q^3 - 1) \cdots (q^n - 1) \\
|\text{SU}_n(q)| &= q^{\frac{1}{2}n(n-1)}(q^2 - 1)(q^3 + 1) \cdots (q^n - (-1)^n) \\
&\vdots \\
|E_8(q)| &= q^{120}(q^2 - 1)(q^8 - 1)(q^{12} - 1)(q^{14} - 1) \cdot (q^{18} - 1)(q^{20} - 1)(q^{24} - 1)(q^{30} - 1)
\end{align*}
$$
**Theorem (Deligne–Lusztig, Lusztig).** Let $k = \mathbb{C}$.

There is a finite set of polynomials $\mathcal{P} \subseteq \mathbb{Q}[X]$, depending only on the "type" of the series $S$, such that

\[
\text{Dim}_\mathbb{C} G(q) \subseteq \{ f(q) \mid f \in \mathcal{P} \} \quad \text{for any } q.
\]

The polynomials in $\mathcal{P}$ may be called “character degree polynomials”; they are explicitly known for each “type”.

**Sketch of proof.** $G(q) = G^F$ where $G$ algebraic group with Weyl group $W$ and $F : G \to G$ Frobenius map. Using algebraic geometry:

\[
(T, \theta) \mapsto R_{T, \theta}
\]

where:

- $T \subseteq G$ any $F$-stable maximal torus, any $\theta \in \text{Irr}_\mathbb{C}(T^F)$;
- $R_{T, \theta}$ integral linear combination of $\text{Irr}_\mathbb{C}(G(q))$. 
1. $\pm R_{T,\theta} \in \text{Irr}_C(G(q))$ if $\theta$ in “general position”; (“most” irreducible representations of $G(q)$ arise in this way).

2. In general, $\langle R_{T,\theta}, R_{T,\theta} \rangle \leq |W|$ and, given $\rho \in \text{Irr}_C(G(q))$, we have $\langle R_{T,\theta}, \rho \rangle \neq 0$ for some $(T, \theta)$.

3. $\pm \dim R_{T,\theta} = [G^F : T^F]_{q^t} = f_w(q)$ where $T$ is of type $w \in W$ and $f_w \in \mathbb{Z}[X]$ only depends on $w$. Hence, $\{f_w \mid w \in W\} \subseteq \mathcal{P}$.

4. For any $\rho \in \text{Irr}_C(G(q))$:

$$\dim \rho = \sum_{(T, \theta) / \sim} \frac{\langle R_{T,\theta}, \rho \rangle}{\langle R_{T,\theta}, R_{T,\theta} \rangle} \dim R_{T,\theta}.$$ 

$$\sim \quad \mathcal{P} \subseteq \left\{ \sum_{w \in W} \frac{n_w}{m_w} f_w \mid n_w, m_w \in \mathbb{Z} \text{ such that } |n_w|, |m_w| \leq |W| \right\}.$$
Consider $S = \{ \text{SL}_2(q) \mid q \text{ prime power} \}$. Let $\ell$ be a prime and $k = \overline{F}_\ell$. Assume first that $q = \ell^f$ (“natural” characteristic); then

$$\{1, \ell, \ell^2, \ell^3, \ldots, \ell^f\} \subseteq \text{Dim}_{\overline{F}_\ell} \text{SL}_2(q)$$

(hence, $\text{Dim}_{\overline{F}_\ell} \text{SL}_2(q)$ can not be described by polynomials in $q$ !)

<table>
<thead>
<tr>
<th>dim $\rho$</th>
<th>Decomposition mod $\ell$, $\ell \nmid q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>irreducible</td>
</tr>
<tr>
<td>$q$</td>
<td>\begin{cases} irreducible &amp; \text{if } \ell \nmid q + 1 \ 1\text{-dim } \oplus (q-1)\text{-dim} &amp; \text{if } \ell \mid q + 1. \end{cases}</td>
</tr>
<tr>
<td>$q \pm 1$</td>
<td>irreducible</td>
</tr>
<tr>
<td>$\frac{1}{2}(q \pm 1)$</td>
<td>irreducible</td>
</tr>
</tbody>
</table>

If $\ell \nmid q$, then $\text{Dim}_{\overline{F}_\ell} \text{SL}_2(q) \subseteq \text{Dim}_\mathbb{C} \text{SL}_2(q)$

(hence, $\text{Dim}_{\overline{F}_\ell} \text{SL}_2(q)$ can be described by polynomials in $q$).
Consider $S = \{ \text{GL}_4(q) \mid q \text{ prime power} \}$.

$$\text{Dim}_\mathbb{C} \text{GL}_4(q) = \{1, q(q^2+q+1), q^2(q^2+1), q^3(q^2+q+1), q^6, \ldots \}$$

<table>
<thead>
<tr>
<th>dim $\rho$</th>
<th>Decomposition mod $\ell$, $\ell \mid q + 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>irreducible</td>
</tr>
<tr>
<td>$q(q^2+q+1)$</td>
<td>1-dim $\oplus (q^3+q^2+q−1)$-dim</td>
</tr>
<tr>
<td>$q^2(q^2+1)$</td>
<td>$(q^3+q^2+q−1)$-dim $\oplus (q−1)^2(q^2+q+1)$-dim</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>

Note: $f_0 := X^3+X^2+X−1 \in \mathbb{Q}[X]$ is irreducible and not a character degree polynomial for $S$ ! However, one can check:

If $\ell \nmid q$, then $\text{Dim}_{\overline{\mathbb{F}}_\ell} \text{GL}_4(q) \subseteq \text{Dim}_\mathbb{C} \text{GL}_4(q) \cup \{f_0(q)\}$.

**Theorem** \((k = \mathbb{C}) / \text{Conjecture.}\)

There is a **finite** set of polynomials \(\mathcal{P} \subseteq \mathbb{Q}[X]\), depending only on the “type” of the series \(S\), such that

\[
\dim_{\mathbb{F}_\ell} G(q) \subseteq \{ f(q) \mid f \in \mathcal{P} \}
\]

for any \(q, \ell\) such that \(\ell \nmid q\).

(Note: If \(\ell \nmid |G(q)|\), then \(\text{Irr}_{\mathbb{F}_\ell}(G(q))\) can be identified with \(\text{Irr}_{\mathbb{C}}(G(q))\) so the conjecture also covers the theorem for \(k = \mathbb{C}\).)

**Status of the conjecture:**

- Known for \(\text{GL}_n(q)\), all \(q, \ell\) (Dipper–James);
- also for \(\text{SU}_3(q), \text{Sp}_4(q)\), all \(q, \ell\) (G., White, Okuyama–Waki).
- For other types of groups, with some restrictions on \(q, \ell\):
  - \(G(q)\) of classical type, \(q \mod \ell\) has odd order (Gruber–Hiss).
Let $D_{q,\ell}$ be the $\ell$-modular decomposition matrix of $G(q)$, $\ell \nmid q$:
(Rows labelled by $\text{Irr}_\mathbb{C}(G(q))$, columns labelled by $\text{Irr}_{\mathbb{F}_\ell}(G(q))$.)

$$D_{q,\ell} = \begin{bmatrix}
D_1 & 0 & \cdots & 0 \\
D_2 & \ddots & \ddots & \vdots \\
B_{q,\ell} & \ddots & \ddots & 0 \\
0 & \cdots & 0 & D_r \\
\ast & \cdots & \cdots & \ast
\end{bmatrix}$$

G.–Hiss (assuming $\ell$ is “good”):

$\leadsto \quad \text{Dim}_\mathbb{C} G(q) + D_{q,\ell} \Rightarrow \text{Dim}_{\mathbb{F}_\ell} G(q)$

Each $D_i$ is square and $\det(D_i) = \pm 1$

Broué–Michel:

$\leadsto \quad \text{Size of } D_i \text{'s bounded by } |W|$

(Total size and number of $D_i$'s grow with $q$)

“Mod $\ell$ Boundedness” Conjecture.

There exists a constant $N$, depending only on the “type” of the series $S$, such that the $\ell$-modular decomposition numbers of $G(q) \in S$, for any $q, \ell$ such that $\ell \nmid q$, are bounded above by $N$. 

Meinolf Geck (Aberdeen University)  
Generic representations  
Lausanne, June 2010 10 / 19
There is a distinguished diagonal block of $D_{q,\ell}^\circ$ (where $\ell$ is “good”):

**“Unipotent representations”**.

Let $D_{q,\ell}^{\text{unip}}$ be the part of $D_{q,\ell}$ with rows and columns labelled by:

$$\text{Unip}_\mathbb{C}(G(q)) = \{ \rho \in \text{Irr}_{\mathbb{C}}(G(q)) \mid \langle R_T, 1, \rho \rangle \neq 0 \text{ for some } T \},$$

$$\text{Unip}_{\mathbb{F}_\ell}(G(q)) = \left\{ \beta \in \text{Irr}_{\mathbb{F}_\ell}(G(q)) \mid \beta \text{ is an } \ell\text{-modular constituent of some } \rho \in \text{Unip}_\mathbb{C}(G(q)) \right\}.$$

Broué–Michel + G.–Hiss $\implies$ $D_{q,\ell}^{\text{unip}}$ is a square diagonal block of $D_{q,\ell}^\circ$.

**Bonnafé–Rouquier (2003)**

The determination of $D_{q,\ell}^\circ$ can almost completely be reduced to the determination of $D_{q,\ell}^{\text{unip}}$ for $G(q)$ and for “smaller” groups $H(q)$.

Reduction is complete for $\text{GL}_n(q)$ (Dipper–James) and $\text{GU}_n(q)$. 
Lusztig’s classification of $\text{Unip}_C(G(q))$.

There exist a finite set $\bar{X}$ and integers $\{m_{\lambda,w} \mid \lambda \in \bar{X}, w \in W\}$, which both only depend on the “type” of the series $S$, such that for any $q$:

1. There exists a bijection $\bar{X} \xrightarrow{1-1} \text{Unip}_C(G(q)), \lambda \mapsto \rho_\lambda$;
2. $\langle R_T, 1, \rho_\lambda \rangle = m_{\lambda,w}$ for $\lambda \in \bar{X}, w \in W$ with $T$ of type $w$.

(The bijection in (1) is almost completely determined by the multiplicities in (2).

From (1) and (2) we obtain the degree polynomials for all $\rho \in \text{Unip}_C(G(q))$.

$$D_{q,\ell}^{\text{unip}} = \begin{bmatrix} \ast & \ast \\ \ast & ? & \ast \\ \ast & \ast & \ast \end{bmatrix} \quad \text{(size } |\bar{X}| \times |\bar{X}|).$$

$\rightsquigarrow$ Principal test case for the “Mod $\ell$ Boundedness Conjecture”.

Meinolf Geck (Aberdeen University)
Examples of matrices $D_{q,\ell}^{\text{unip}}$ (rows labelled by $\text{Unip}_\mathbb{C}(G(q)) \leftrightarrow \bar{X}$):

<table>
<thead>
<tr>
<th>$GL_3(q)$</th>
<th>$\ell \mid q+1$</th>
<th>$GU_3(q)$</th>
<th>$2\neq \ell \mid q+1$</th>
<th>$GL_4(q)$</th>
<th>$2 \neq \ell \mid q+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td>1 . . .</td>
<td>(3)</td>
<td>1 . . .</td>
<td>(4)</td>
<td>1 . . . .</td>
</tr>
<tr>
<td>(21)</td>
<td>. 1 .</td>
<td>(21)</td>
<td>. 1 .</td>
<td>(31)</td>
<td>1 1 . .</td>
</tr>
<tr>
<td>(111)</td>
<td>1 . 1</td>
<td>(111)</td>
<td>1 2 2 1</td>
<td>(22)</td>
<td>. 1 1 .</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\rho_{(21)}$ cuspidal</td>
<td>(211)</td>
<td>1 1 1 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(1111)</td>
<td>1 . . 1 1</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$Sp_4(q)$</th>
<th>$2, 3 \neq \ell \mid q+1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1_G$</td>
<td>1 . . . . . .</td>
</tr>
<tr>
<td>$\theta_{10}$</td>
<td>. 1 . . . . .</td>
</tr>
<tr>
<td>$\varepsilon_1$</td>
<td>1 . 1 . . .</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>1 . . 1 . . .</td>
</tr>
<tr>
<td>refl</td>
<td>. . . 1 .</td>
</tr>
<tr>
<td>$St_G$</td>
<td>1 2 2 1 1</td>
</tr>
</tbody>
</table>

$\text{Irr}(W) = \{1_W, \varepsilon_1, \varepsilon_2, \varepsilon, \text{refl}\}$,

$\theta_{10}$ cuspidal (Srinivasan)

Sources:

$GL_3, GL_4$ : James;

$GU_3$ : G., Okuyama–Waki (2002);

$Sp_4$ : White, Okuyama–Waki.

2 difficult to determine!
Which ordering of $\text{Unip}_{\mathbb{C}}(G(q))$ (the rows of $D_{q,\ell}^\text{unip}$)?

- First approach: Order by dim $\rho$. (Works in above examples.)
- Order “generically”: Use degrees of degree polynomials.

Dipper–James $\leadsto$ “proper” partial order for $\text{GL}_n(q)$.

- $\text{Unip}_{\mathbb{C}}(\text{GL}_n(q)) = \{ \rho_\lambda \mid \lambda \text{ partition of } n \}$.
- Dipper–James use dominance order $\preceq$:
  \[ \lambda \preceq \mu \iff \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq \mu_1 + \mu_2 + \cdots + \mu_d \text{ for all } d \geq 1. \]
- Observation: Consider unipotent classes of $\text{GL}_n(\mathbb{C})$ (matrices with all eigenvalues equal to 1).
  - $\{ O_\lambda \mid \lambda \text{ partition of } n \}$ (Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots$).
  - $O_\lambda \subseteq \overline{O}_\mu$ (Zariski closure) $\iff$ $\lambda \preceq \mu$. 

Meinolf Geck (Aberdeen University)
Character values on unipotent classes:

<table>
<thead>
<tr>
<th>GL(_3(q))</th>
<th>(111)</th>
<th>(21)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td>1</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>(21)</td>
<td>(q(q+1))</td>
<td>(qq)</td>
<td>.</td>
</tr>
<tr>
<td>(111)</td>
<td>(q^3q^3)</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>GU(_3(q))</th>
<th>(111)</th>
<th>(21)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3)</td>
<td>1</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>(21)</td>
<td>(q(q−1))</td>
<td>(−q−q)</td>
<td>.</td>
</tr>
<tr>
<td>(111)</td>
<td>(q^3q^3)</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Sp(_4(q))</th>
<th>(1111)</th>
<th>(211)</th>
<th>(22)</th>
<th>(22)'</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1(_G)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>(θ_{10})</td>
<td>(\frac{1}{2}q(q−1)^2)</td>
<td>(-\frac{1}{2}q(q−1))</td>
<td>.</td>
<td>(qq)</td>
<td>.</td>
</tr>
<tr>
<td>(ε_1)</td>
<td>(\frac{1}{2}q(q^2+1))</td>
<td>(\frac{1}{2}q(q+1))</td>
<td>.</td>
<td>(qq)</td>
<td>.</td>
</tr>
<tr>
<td>(ε_2)</td>
<td>(\frac{1}{2}q(q^2+1))</td>
<td>(-\frac{1}{2}q(q−1))</td>
<td>(qq)</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>refl</td>
<td>(\frac{1}{2}(q+1)^2)</td>
<td>(\frac{1}{2}q(q+1))</td>
<td>(qq)</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>St(_G)</td>
<td>(q^4q^4)</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

Idea: Associate to each \(ρ\) a unipotent class of \(G\) where \(G(q) = G^F\).
Write \( G(q) = G^F \) and let \( O \subseteq G \) be an \( F \)-stable unipotent class. Let \( u \in O^F \) and \( A(u) := C_G(u)/C_G(u)^\circ \). Then define:

\[
AV(\rho, O) := \sum_{a \in A(u)} \text{trace}(u_a, \rho) \quad \text{for } \rho \in \text{Irr}_C(G(q)).
\]

(To \( a \in A(u) \) one can associate \( u_a \in O^F \), well-defined up to conjugacy in \( G(q) \).)

**Lusztig’s (+ G.–Malle’s) “Unipotent Support”.

Let \( \rho \in \text{Irr}_C(G(q)) \). Then there is a unique \( F \)-stable unipotent class in \( G \), which we shall denote by \( O_\rho \), such that

\[
AV(\rho, O_\rho) \neq 0 \quad \text{and} \quad AV(\rho, O) = 0 \quad \text{if } \dim O < \dim O_\rho.
\]

(There are examples where \( \text{trace}(u, \rho) \neq 0 \) with \( u \in O \) and \( \dim O > \dim O_\rho \).)

\[
AV(\rho, O_\rho) = \pm n_\rho^{-1} |A(u)| q^{\dim B_u},
\]

where \( u \in O_\rho \) and \( n_\rho \) is a “small” integer \( \leq |W| \).
Re-consider the example $G(q) = \text{Sp}_4(q)$:

<table>
<thead>
<tr>
<th>$\text{Sp}_4(q)$</th>
<th>(1111)</th>
<th>(211)</th>
<th>(22)</th>
<th>(22)'</th>
<th>(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1$_G$</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>*</td>
<td>1</td>
</tr>
<tr>
<td>$\theta_{10}$</td>
<td>*</td>
<td>*</td>
<td>.</td>
<td>$q$</td>
<td>.</td>
</tr>
<tr>
<td>$\varepsilon_1$</td>
<td>*</td>
<td>*</td>
<td>.</td>
<td>$q$</td>
<td>.</td>
</tr>
<tr>
<td>$\varepsilon_2$</td>
<td>*</td>
<td>*</td>
<td>$q$</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>refl</td>
<td>*</td>
<td>*</td>
<td>$q$</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>$\text{St}_G$</td>
<td>$q^4$</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

$((22), (22)'$ conjugate in $G$)

|                      | $2, 3 \neq \ell | q + 1$ | $2, 3$ |
|----------------------|-------------------|--------|
| 1                    | . . . . . .        | 1      |
| . 1                  | . . . . . .        | 1      |
| 1 . 1 . . . . 1      | 1 . . . . . . .    |        |
| 1 . . 1 . . . 1      | . . . . 1 . . .    |        |
| 1 2 1 1 1 . 1 1 1 2  |                    |        |

Observation:

Three identity blocks in $D_{q, \ell}^{\text{unip}}$ corresponding to the three possible unipotent supports.
Recall $\text{Unip}_C(G(q)) = \{\rho_\lambda \mid \lambda \in \bar{X}\}$. Write $O_\lambda := O_{\rho_\lambda}$ for $\lambda \in \bar{X}$.

$$\bar{X} \rightarrow \{ \text{F-stable unipotent classes in } G \}, \quad \lambda \mapsto O_\lambda.$$ 

“Strong Triangularity” Conjecture (G., Hiss). Assume that $\ell$ is “good”.

There exists a labelling $\text{Unip}_{\overline{\mathbb{F}_\ell}}(G(q)) = \{\beta_\mu \mid \mu \in \bar{X}\}$ such that:

1. $[\rho_\lambda : \beta_\lambda] = 1$ for all $\lambda \in \bar{X}$;
2. $[\rho_\lambda : \beta_\mu] = 0$ unless $\lambda = \mu$ or $O_\lambda \subsetneq \overline{O}_\mu$ (Zariski closure).

(Note: If such a labelling exists, then it is uniquely determined.)

True for: $\text{GL}_n(q)$ (Dipper–James, $q$-Schur algebra), $G_2(q)$ (Hiss, explicit computations), $\text{GU}_n(q)$ (G., generalised Gelfand–Graev representations), and some further cases of small rank, $\text{Sp}_4(q)$, $^3D_4(q)$, ....
Theorem (G. 2007/09). Assume $G(q)$ “split” (that is, $\gamma = \text{id}$).

The “Strong Triangularity” Conjecture is true as far as decomposition numbers of “principal series” representations are concerned. (That is, those $\rho \in \text{Unip}_\mathbb{C}(G(q))$ and $\beta \in \text{Unip}_{\mathbb{F}_\ell}(G(q))$ which admit non-zero vectors fixed by a Borel subgroup $B(q) \subseteq G(q)$.)

By Dipper, the part of $D_{q,\ell}^{\text{unip}}$ corresponding to principal series representations is determined by the decomposition matrix of the Iwahori–Hecke algebra $H_q(W)$ associated with $G(q)$.

Then the proof of the theorem relies on:

2. Characterisation in terms of order on unipotent classes (2009).