Basic sets for Hecke algebras

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A complex reflection group $W$ is a finite group of matrices with coefficients in a finite abelian extension $K$ of $\mathbb{Q}$ generated by pseudo-reflections.

If $K = \mathbb{Q}$, then $W$ is a Weyl group.

Shephard-Todd classification (1954)

The irreducible complex reflection groups are:

- the groups of the infinite series $G(de, e, r)$
  
  (with $G(d, 1, r) \cong \mathbb{Z}/d\mathbb{Z} \rtimes S_r$);

- the exceptional groups $G_4, G_5, \ldots, G_{37}$. 

Hecke algebras of complex reflection groups

Let $W$ be a complex reflection group.

The group $W$ has a presentation given by:

- generators: $S$
- relations:
  - braid relations;
  - $s^{es} = 1$.

Example:

$$G := G(3, 1, 2) = \langle s, t \mid stst = tsts, s^3 = 1, t^2 = 1 \rangle.$$
Let $q$ be an indeterminate and let $A := \mathbb{Z}_K[q, q^{-1}]$.

The cyclotomic Hecke algebra $\mathcal{H}_q(W)$ has a presentation given by:

- generators: $(T_s)_{s \in S}$
- relations:
  - braid relations;
  - $(T_s - q^{m_s,0})(T_s - \zeta_{e_s}q^{m_s,1}) \cdots (T_s - \zeta_{e_s}^{-1}q^{m_s,e_s-1}) = 0$.

**Example:** $G = G(3, 1, 2)$

\[
\mathcal{H}_q(G) = \left\langle T_s, T_t \right| \begin{align*}
T_s T_t T_s T_t &= T_t T_s T_t T_s, \\
(T_s - q^{m_s,0})(T_s - \zeta_3q^{m_s,1})(T_s - \zeta_3^2q^{m_s,2}) &= 0, \\
(T_t - q^{m_t,0})(T_t + q^{m_t,1}) &= 0
\end{align*}\]
Schur elements of Hecke algebras

Assumptions

1. The algebra $\mathcal{H}_q(W)$ is a free $A$-module of rank $|W|$.
2. There exists a “canonical” symmetrizing form $t : \mathcal{H}_q(W) \to A$.
3. The algebra $K(q)\mathcal{H}_q(W)$ is split.

The algebra $K(q)\mathcal{H}_q(W)$ is also semisimple. By Tits’ deformation theorem, there exists a bijection

$$\text{Irr}(K(q)\mathcal{H}_q(W)) \leftrightarrow \text{Irr}(W)$$

$$\chi_q \mapsto \chi.$$

Moreover, we have

$$t = \sum_{\chi \in \text{Irr}(W)} \frac{1}{s_{\chi}} \chi_q$$

where $s_{\chi}$ is the Schur element of $\mathcal{H}_q(W)$ associated to $\chi$. 
All Schur elements \( s_\chi \) belong to \( A = \mathbb{Z}_K[q, q^{-1}] \) and they are products of \( K \)-cyclotomic polynomials.

We can define the following three functions on \( \text{Irr}(W) \):

- \( a : \text{Irr}(W) \rightarrow \mathbb{Z}, \chi \mapsto \text{valuation}(s_\chi) \);
- \( A : \text{Irr}(W) \rightarrow \mathbb{Z}, \chi \mapsto \text{degree}(s_\chi) \);
- \( c : \text{Irr}(W) \rightarrow \mathbb{Z}, \chi \mapsto a_\chi + A_\chi \).

**Example:**

If \( s_\chi = q^{-1} + 2 + q \), then \( a_\chi = -1 \), \( A_\chi = 1 \) and \( c_\chi = 0 \).
The decomposition matrix

Let

\[ \theta : A \to L, \quad q \mapsto \xi \]

be a ring homomorphism such that \( L \) is the field of fractions of \( \theta(A) \).

Assume that \( L\mathcal{H}_q \) is split. Let \( R_0(K(q)\mathcal{H}_q) \) and \( R_0(L\mathcal{H}_q) \) be the Grothendieck groups of finitely generated \( K(q)\mathcal{H}_q \)-modules and \( L\mathcal{H}_q \)-modules respectively.

We have a well-defined \textit{decomposition map}

\[ d_\theta : R_0(K(q)\mathcal{H}_q) \to R_0(L\mathcal{H}_q) \]

with corresponding \textit{decomposition matrix}

\[ D_\theta = ( [E : M] )_{E \in \text{Irr}(W), \ M \in \text{Irr}(L\mathcal{H}_q)} \cdot \]
Basic sets

**Definition (Geck-Rouquier)**

We say that $\mathcal{H}_q$ admits a **canonical basic set** $B^a \subset \text{Irr}(W)$ with respect to $\theta : A \to L$ if and only if the following two conditions are satisfied:

1. For all $M \in \text{Irr}(L\mathcal{H}_q)$, there exists $E_M \in B^a$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $a_E > a_{E_M}$.

2. The map

\[ \text{Irr}(L\mathcal{H}_q) \to B^a \]

\[ M \mapsto E_M \]

is a bijection.
If $\mathcal{H}_q$ admits a canonical basic set $B^a$ with respect to $\theta$, then the decomposition matrix $D_\theta$ has the following form:

$$D_\theta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
* & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
* & * & \cdots & 1 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
\end{pmatrix} \left\{ \begin{array}{c}
\text{Irr}(L\mathcal{H}_q) \\
B^a \\
\text{Irr}(W)
\end{array} \right\}$$
From now on, $L$ will be a field of characteristic zero.

**Theorem (Geck-Rouquier, Geck, Geck-Jacon, C.-Jacon)**

Let $W$ be a Weyl group. The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

**Theorem (Dipper-James-Murphy, Geck-Rouquier, Ariki, Uglov, Jacon)**

Let $W$ be a complex reflection group of type $G(d,1,r)$. The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

**Theorem (Genet-Jacon)**

Let $W$ be a complex reflection group of type $G(de,e,r)$. The algebra $\mathcal{H}_q(W)$ for a certain choice of parameters admits a canonical basic set with respect to any specialization $\theta : A \to L$. 
Proposition (Ginzburg-Guay-Opdam-Rouquier, C.-Gordon)

Let $W$ be any complex reflection group. For every specialization $\theta : A \to L$, there exists a subset $B^c \subset \text{Irr}(W)$ such that:

1. For all $M \in \text{Irr}(LH_q)$, there exists $E_M \in B^c$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $c_E > c_{E_M}$.

2. The map
   \[
   \begin{array}{ccc}
   \text{Irr}(LH_q) & \to & B^c \\
   M & \mapsto & E_M
   \end{array}
   \]
   is a bijection.

Corollary

Let $W$ be a complex reflection group of type $G(d, 1, r)$. The algebra $H_q(W)$ admits a canonical basic set $B^a \subset \text{Irr}(W)$ with respect to any specialization $\theta : A \to L$. 
Proposition (C.-Miyachi)

Let $W$ be an exceptional complex reflection group of rank 2 whose Hecke algebra $H_q(W)$ appears in the “cyclotomic Harish-Chandra series” ($W \in \{ G_4, G_5, G_8, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22} \}$). The algebra $H_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

Moreover, there exists a subset $B^{opt} \subset \text{Irr}(W)$ such that:

1. For all $M \in \text{Irr}(LH_q)$, there exists $E_M \in B^{opt}$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $E \notin B^{opt}$.

2. The map

$$\begin{align*}
\text{Irr}(LH_q) & \to B^{opt} \\
M & \mapsto E_M
\end{align*}$$

is a bijection.
If $W$ is as above, then the decomposition matrix $D_\theta$ has the following form:

$$D_\theta = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix}$$

\[ \begin{cases}
B_{\text{opt}} \\
\text{Irr}(W)
\end{cases} \]

\[ \begin{cases}
\text{Irr}(LH_q)
\end{cases} \]
ΧΡΟΝΙΑ ΠΟΛΛΑ, JACQUES!!!