Basic sets for Hecke algebras

Maria Chlouveraki

University of Edinburgh

Group Representation Theory and Related Topics

Conférence en l'honneur de Jacques Thévenaz

June 23, 2010

Complex reflection groups

A complex reflection group W is a finite group of matrices with coefficients in a finite abelian extension K of $\mathbb{Q}$ generated by *pseudo-reflections*.

If $K = \mathbb{Q}$, then W is a Weyl group.

Shephard-Todd classification (1954)

The irreducible complex reflection groups are:

- the groups of the infinite series $G(de, e, r)$

  (with $G(d, 1, r) \cong \mathbb{Z}/d\mathbb{Z} \wr \mathfrak{S}_r$);
- the exceptional groups G_4, G_5, $\ldots$, G_{37}.

Hecke algebras of complex reflection groups

Let W be a complex reflection group.

The group W has a presentation given by:

- generators: S
- relations:
  - braid relations;
  - $s^{e_s} = 1$.

Example:

$$G := G(3,1,2) = \langle\, s, t \mid stst = tsts,\ s^3 = 1,\ t^2 = 1 \,\rangle.$$

Let q be an indeterminate and let $A := \mathbb{Z}_K[q, q^{-1}]$.

The cyclotomic Hecke algebra $\mathcal{H}_q(W)$ has a presentation given by:

- generators: $(T_s)_{s \in S}$
- relations:
  - braid relations;
  - $(T_s - 1q^{m_{s,0}})(T_s - \zeta_{e_s} q^{m_{s,1}}) \cdots (T_s - \zeta_{e_s}^{e_s - 1} q^{m_{s,e_s-1}}) = 0.$

Example: $G = G(3,1,2)$

$$\mathcal{H}_q(G) = \left\langle \; T_s, T_t \;\middle|\; \begin{array}{l} T_s T_t T_s T_t = T_t T_s T_t T_s, \\ (T_s - q^{m_{s,0}})(T_s - \zeta_3 q^{m_{s,1}})(T_s - \zeta_3^2 q^{m_{s,2}}) = 0, \\ (T_t - q^{m_{t,0}})(T_t + q^{m_{t,1}}) = 0 \end{array} \right\rangle.$$

Schur elements of Hecke algebras

Assumptions

1. The algebra $\mathcal{H}_q(W)$ is a free A-module of rank $|W|$.
2. There exists a "canonical" symmetrizing form $t : \mathcal{H}_q(W) \to A$.
3. The algebra $K(q)\mathcal{H}_q(W)$ is split.

The algebra $K(q)\mathcal{H}_q(W)$ is also semisimple. By Tits' deformation theorem, there exists a bijection

$$\begin{array}{ccc} \mathrm{Irr}(K(q)\mathcal{H}_q(W)) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_q & \mapsto & \chi. \end{array}$$

Moreover, we have

$$t = \sum_{\chi \in \mathrm{Irr}(W)} \frac{1}{s_\chi} \chi_q$$

where s_χ is the Schur element of $\mathcal{H}_q(W)$ associated to χ.

All Schur elements s_χ belong to $A = \mathbb{Z}_K[q, q^{-1}]$ and they are products of K-cyclotomic polynomials.

We can define the following three functions on $\mathrm{Irr}(W)$:

- $a\ : \mathrm{Irr}(W) \to \mathbb{Z}$, $\chi \mapsto \mathrm{valuation}(s_\chi)$;
- $A : \mathrm{Irr}(W) \to \mathbb{Z}$, $\chi \mapsto \mathrm{degree}(s_\chi)$;
- $c\ : \mathrm{Irr}(W) \to \mathbb{Z}$, $\chi \mapsto a_\chi + A_\chi$.

Example:

If $s_\chi = q^{-1} + 2 + q$, then $a_\chi = -1$, $A_\chi = 1$ and $c_\chi = 0$.

The decomposition matrix

Let

$$\theta : A \to L, \quad q \mapsto \xi$$

be a ring homomorphism such that L is the field of fractions of $\theta(A)$.

Assume that $L\mathcal{H}_q$ is split. Let $R_0(K(q)\mathcal{H}_q)$ and $R_0(L\mathcal{H}_q)$ be the Grothendieck groups of finitely generated $K(q)\mathcal{H}_q$-modules and $L\mathcal{H}_q$-modules respectively.

We have a well-defined decomposition map

$$d_\theta : R_0(K(q)\mathcal{H}_q) \to R_0(L\mathcal{H}_q)$$

with corresponding decomposition matrix

$$D_\theta = (\,[E : M]\,)_{E \in \mathrm{Irr}(W),\, M \in \mathrm{Irr}(L\mathcal{H}_q)}\,.$$

Basic sets

Definition (Geck-Rouquier)

We say that $\mathcal{H}_q$ admits a canonical basic set $\mathcal{B}^a \subset \mathrm{Irr}(W)$ with respect to $\theta : A \to L$ if and only if the following two conditions are satisfied:

1. For all $M \in \mathrm{Irr}(L\mathcal{H}_q)$, there exists $E_M \in \mathcal{B}^a$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $a_E > a_{E_M}$.
2. The map

$$\begin{array}{ccc} \mathrm{Irr}(L\mathcal{H}_q) & \to & \mathcal{B}^a \\ M & \mapsto & E_M \end{array}$$

   is a bijection.

If $\mathcal{H}_q$ admits a canonical basic set $\mathcal{B}^a$ with respect to θ, then the decomposition matrix D_θ has the following form:

$$D_\theta = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\mathrm{Irr}(L\mathcal{H}_q)} \begin{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \mathcal{B}^a \\ \vphantom{\begin{matrix}1\\1\\1\end{matrix}} \end{matrix} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\\1\end{matrix}}\right\} \mathrm{Irr}(W)$$

From now on, L will be a field of characteristic zero.

Theorem (Geck-Rouquier, Geck, Geck-Jacon, C.-Jacon)

Let W be a Weyl group. The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

Theorem (Dipper-James-Murphy, Geck-Rouquier, Ariki, Uglov, Jacon)

Let W be a complex reflection group of type $G(d, 1, r)$. The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

Theorem (Genet-Jacon)

Let W be a complex reflection group of type $G(de, e, r)$. The algebra $\mathcal{H}_q(W)$ for a certain choice of parameters admits a canonical basic set with respect to any specialization $\theta : A \to L$.

Proposition (Ginzburg-Guay-Opdam-Rouquier, C.-Gordon)

Let W be any complex reflection group. For every specialization $\theta : A \to L$, there exists a subset $\mathcal{B}^c \subset \mathrm{Irr}(W)$ such that:

1. For all $M \in \mathrm{Irr}(L\mathcal{H}_q)$, there exists $E_M \in \mathcal{B}^c$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $c_E > c_{E_M}$.
2. The map

$$\begin{array}{ccc} \mathrm{Irr}(L\mathcal{H}_q) & \to & \mathcal{B}^c \\ M & \mapsto & E_M \end{array}$$

   is a bijection.

Corollary

Let W be a complex reflection group of type $G(d, 1, r)$. The algebra $\mathcal{H}_q(W)$ admits a canonical basic set $\mathcal{B}^a \subset \mathrm{Irr}(W)$ with respect to any specialization $\theta : A \to L$.

Proposition (C.-Miyachi)

Let W be an exceptional complex reflection group of rank 2 whose Hecke algebra $\mathcal{H}_q(W)$ appears in the "cyclotomic Harish-Chandra series" ($W \in \{G_4, G_5, G_8, G_9, G_{10}, G_{12}, G_{16}, G_{20}, G_{22}\}$). The algebra $\mathcal{H}_q(W)$ admits a canonical basic set with respect to any specialization $\theta : A \to L$.

Moreover, there exists a subset $\mathcal{B}^{\mathrm{opt}} \subset \mathrm{Irr}(W)$ such that:

1. For all $M \in \mathrm{Irr}(L\mathcal{H}_q)$, there exists $E_M \in \mathcal{B}^{\mathrm{opt}}$ such that
   - $[E_M : M] = 1$, and
   - if $[E : M] \neq 0$, then either $E = E_M$ or $E \notin \mathcal{B}^{\mathrm{opt}}$.
2. The map

$$\begin{array}{ccc} \mathrm{Irr}(L\mathcal{H}_q) & \to & \mathcal{B}^{\mathrm{opt}} \\ M & \mapsto & E_M \end{array}$$

   is a bijection.

If W is as above, then the decomposition matrix D_θ has the following form:

$$D_\theta = \underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}}_{\mathrm{Irr}(L\mathcal{H}_q)} \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \mathcal{B}^{\mathrm{opt}} \\ \\ \\ \\ \end{matrix} \left.\begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}\right\} \mathrm{Irr}(W)$$

ΧΡΟΝΙΑ ΠΟΛΛΑ, JACQUES!!!