Separability and descent in triangulated categories

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Outline

Basic definitions

A *tensor triangulated category* $\mathcal{K}$ is an additive category $\mathcal{K}$ with

- a *suspension* $\Sigma : \mathcal{K} \xrightarrow{\sim} \mathcal{K}$ and distinguished *exact triangles* in $\mathcal{K}$

$$a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} \Sigma a$$

  satisfying some very natural axioms

- *monoidal structure* $\mathcal{K} \times \mathcal{K} \xrightarrow{\otimes} \mathcal{K}$ symmetric ($a \otimes b \cong b \otimes a$), with unit $1 \in \mathcal{K}$ ($1 \otimes a = a$), compatible with triangulation ($a \otimes -$ exact).

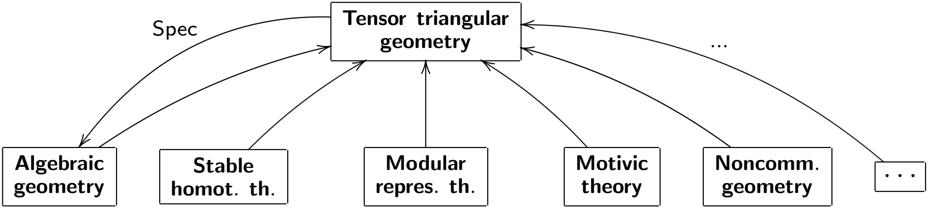
Example: Perfect complexes

$\mathrm{D}^{\mathrm{perf}}(X)$ for X scheme; *e.g.* R commutative, $\mathrm{D}^{\mathrm{perf}}(R) = \mathrm{K}^{\mathrm{b}}(R\text{–proj})$.

Example: Modular representation theory

$$kG\text{–stab} := \frac{kG\text{–mod}}{kG\text{–proj}} \cong \frac{\mathrm{D}^{\mathrm{b}}(kG\text{–mod})}{\mathrm{K}^{\mathrm{b}}(kG\text{–proj})}\,.$$

Big picture



Spec

Tensor triangular geometry

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Algebraic geometry

Stable homot. th.

Modular repres. th.

Motivic theory

Noncomm. geometry

. . .

The spectrum

Let $\mathcal{K}$ $\otimes$-triangulated *essentially small*. There exists a topological space

$$\mathrm{Spc}(\mathcal{K}) = \mathrm{Spc}(\mathcal{K}, \otimes, 1)$$

called the *spectrum* of $\mathcal{K}$ and a choice of a closed subset

$$\mathrm{supp}(a) \subset \mathrm{Spc}(\mathcal{K})$$

for every object $a \in \mathcal{K}$, called the *support* of a, such that

- $\mathrm{supp}(0) = \varnothing$ and $\mathrm{supp}(1) = \mathrm{Spc}(\mathcal{K})$
- $\mathrm{supp}(a \oplus b) = \mathrm{supp}(a) \cup \mathrm{supp}(b)$
- $\mathrm{supp}(\Sigma a) = \mathrm{supp}(a)$
- if $a \to b \to c \to \Sigma a$ is exact then $\mathrm{supp}(c) \subset \mathrm{supp}(a) \cup \mathrm{supp}(b)$
- $\mathrm{supp}(a \otimes b) = \mathrm{supp}(a) \cap \mathrm{supp}(b)$

which is moreover *universal* (final) for these properties.

Computing the spectrum

Determining $\mathrm{Spc}(\mathcal{K})$ is equivalent to classify thick tensor ideals of $\mathcal{K}$. Structure sheaf $\mathcal{O}_{\mathcal{K}}$ on $\mathrm{Spc}(\mathcal{K})$ via $\mathrm{End}_{\mathcal{K}}(1)$ and localization. Hence a locally ringed space $\mathrm{Spec}(\mathcal{K}) = \big(\,\mathrm{Spc}(\mathcal{K}), \mathcal{O}_{\mathcal{K}}\big)$.

Theorem

X quasi-compact and quasi-separated scheme: $\mathrm{Spec}(\mathrm{D}^{\mathrm{perf}}(X)) \cong X$.

Corollary

If $\mathrm{D}^{\mathrm{perf}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(Y)$ *as $\otimes$-triangulated categories then* $X \simeq Y$.

Theorem

G finite group and k field. Then $\mathrm{Spec}(\mathrm{D}^{\mathrm{b}}(kG\!-\!\mathrm{mod})) \cong \mathrm{Spec}^{\mathrm{h}}(\mathrm{H}^{\bullet}(G,k))$ *and* $\mathrm{Spec}(kG\,-\mathrm{stab}) \cong \mathrm{Proj}(\mathrm{H}^{\bullet}(G,k)) = \mathcal{V}_G$ *projective support variety.*

Localization

Key construction

Let $U \subset \mathrm{Spc}(\mathcal{K})$ open with complement $Z = \mathrm{Spc}(\mathcal{K}) - U$. Construct

$$\mathcal{K}(U) := \big(\mathcal{K} \,/\, \{\, a \in \mathcal{K} \,\big|\, \mathrm{supp}(a) \subset Z \,\}\big)^{\natural}.$$

$\mathcal{K}(U)$ is *the category* $\mathcal{K}$ *over* U.

Examples

For $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$ and $U \subset X$, then $\mathcal{K}(U) \cong \mathrm{D}^{\mathrm{perf}}(U)$ by Thomason.
For $\mathcal{K} = kG-$stab and $U \subset \mathcal{V}_G$ non-trivial, $\mathcal{K}(U)$ is never a $kH-$stab !

Theorem (B. 2008)

There exists a monomorphism $\mathrm{Pic}(\mathcal{V}_G) \otimes \mathbb{Z}[1/p] \hookrightarrow T(G) \otimes \mathbb{Z}[1/p]$ *which induces an isomorphism* $\mathrm{Pic}(\mathcal{V}_G) \otimes \mathbb{Q} \cong T(G) \otimes \mathbb{Q}$.

Local triviality of $\otimes$-invertibles

Remark

$T(G) = \mathrm{Pic}(kG{-}\mathrm{stab})$ where $\mathrm{Pic}(\mathcal{K}) = \{a \in \mathcal{K}\,,\ \otimes\text{-invertible}\}/\simeq$.
In $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, $\otimes$-invertible objects are locally trivial.
This fails in general tensor triangular geometry.

Quaternion group

For $G = Q_8$ then $\mathcal{V}_G = *$ and $T(G) = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ (when "$k \ni \sqrt[3]{1}$").

Corollary of $\mathrm{Pic}(\mathcal{V}_G) \otimes \mathbb{Q} \simeq T(G) \otimes \mathbb{Q}$

Any endotrivial M is locally torsion: $\exists\, n \neq 0$ with $M^{\otimes n} \simeq 1$ locally.

Question

What about finite group *schemes*?

Local torsion of $\otimes$-invertibles

Naive question

For a general $\mathcal{K}$, is $\operatorname{Pic}(\mathcal{K})$ locally torsion? Answer : no!

Proposition (B-Virk, 2009)

Let $\mathcal{K}$ $\otimes$*-triangulated and* T *any abelian group. Let* $\mathcal{L} := \coprod_T \mathcal{K}$ $\otimes$*-triangulated. Then* $\operatorname{Spec}(\mathcal{L}) = \operatorname{Spec}(\mathcal{K})$ *but* $\operatorname{Pic}(\mathcal{L}) = \operatorname{Pic}(\mathcal{K}) \times T$.

Example (T. Peter)

Assume k satisfies Beilinson-Soulé Vanishing Conjecture $\mathrm{H}^p(k, \mathbb{Q}(q)) = 0$ for $p \leq 0$ and $q > 0$ (*e.g.* k a number field) then mixed Tate motives $\mathrm{DMT}(k)_{\mathbb{Q}}$ is local ($\mathrm{Spc} = *$) but $\operatorname{Pic}\big(\mathrm{DMT}(k)_{\mathbb{Q}}\big) = \{\mathbb{Q}(i)[j]\} = \mathbb{Z} \times \mathbb{Z}$.

Open problem

Is $\operatorname{Pic}(\mathcal{K})$ locally torsion when $\mathcal{K}$ is generated by 1 ?

"Local" triviality of $\otimes$-invertibles

Standard Grothendieckian idea

Is there a more flexible sense of "local" such that any $\otimes$-invertible $u \in \mathcal{K}$ is "locally" trivial? Say, for some generalized "covers" $\mathcal{K} \longrightarrow \mathcal{L}$.

Proposition

Let $u \in \mathcal{K}$ *$\otimes$-invertible with* $\sigma_{u,u} = \mathrm{id}_{u\otimes u}$ *(e.g.* $u = v^{\otimes 2}$*). Suppose that* $[u] \in \mathrm{Pic}(\mathcal{K})$ *is torsion. Choose* $\alpha : u^{\otimes n} \overset{\sim}{\to} 1$ *(*$n > 0$*) and define*

$$A = \bigoplus_{i\in\mathbb{Z}/n} u^{\otimes i} = 1 \oplus u \oplus u^{\otimes 2} \oplus \cdots \oplus u^{\otimes(n-1)}$$

with (obvious) multiplication via α*. Then the image of* u *in the category of* A*-modules (in* $\mathcal{K}$*) is trivial:* $F_A(u) \simeq 1_A$*.*

Note that A is faithful: $A \otimes f = 0$ then $f = 0$. Some converse holds!

Summary of motivations

- Need to explain "*A*-modules" for the last result.
- That result says that the torsion part of $\mathrm{Pic}(\mathcal{K})$ is locally trivial with respect to "some sort of faithful topology" (finer than Zariski).
- Beyond $\otimes$-invertibles: More constructions for $\otimes$-triangulated categories using "*A*-modules"? For instance
  - For $\mathcal{K} = \mathrm{D}^{\mathrm{perf}}(X)$, X a scheme, recall $\mathcal{K}(U) \cong \mathrm{D}^{\mathrm{perf}}(U)$ when $U \subset X$ open. Can one realize $\mathrm{D}^{\mathrm{perf}}(Z)$ for $Z \hookrightarrow X$ closed subscheme?
  - For $\mathcal{K} = kG-$stab, can one realize $kH-$stab for $H < G$ subgroup?

Rings and monads

Ring object in $(\mathcal{K}, \otimes, 1)$

$A \in \mathcal{K}$ with multiplication $\mu : A \otimes A \to A$ and unit $\eta : 1 \to A$ such that

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "1\otimes\eta"] \arrow[dr, equal] & A \otimes A \arrow[d, "\mu"] & A \arrow[l, "\eta\otimes 1"'] \arrow[dl, equal] \\
& A\,. &
\end{tikzcd}$$

Monad in $\mathcal{K}$

Functor $M : \mathcal{K} \to \mathcal{K}$ with $\mu : M^2 \to M$ and $\eta : \mathrm{Id} \to M$ such that

$$\begin{tikzcd}
M^3 \arrow[r, "M\mu"] \arrow[d, "\mu M"'] & M^2 \arrow[d, "\mu"] \\
M^2 \arrow[r, "\mu"'] & M
\end{tikzcd}
\qquad
\begin{tikzcd}
M \arrow[r, "M\eta"] \arrow[dr, equal] & M^2 \arrow[d, "\mu"] & M \arrow[l, "\eta M"'] \arrow[dl, equal] \\
& M\,. &
\end{tikzcd}$$

Modules over a monad

Modules

(M, μ, η) monad on $\mathcal{K}$. An *M-module* $(x \in \mathcal{K}\,,\ \varrho : M(x) \to x)$ s.t.

$$\begin{tikzcd}
M^2(x) \arrow[r, "M(\varrho)"] \arrow[d, "\mu_x"'] & M(x) \arrow[d, "\varrho"] \\
M(x) \arrow[r, "\varrho"'] & x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, equal] & M(x) \arrow[d, "\varrho"] \\
 & x\,.
\end{tikzcd}$$

Example : *free M-module* $F_M(y) = \big(M(y), \mu_y\big)$ for any $y \in \mathcal{K}$.

Eilenberg-Moore

$M\!-\!\mathrm{Mod}_{\mathcal{K}}$ category of M-modules in $\mathcal{K}$. Adjunction:

$$\begin{tikzcd}
\mathcal{K} \arrow[d, shift right, "F_M"'] \\
M\!-\!\mathrm{Mod}_{\mathcal{K}} \arrow[u, shift right, "U_M"']
\end{tikzcd}\ .$$

Restriction is extension of scalars

Let $H < G$ be a subgroup (of finite index) and k a field.

Ring object $k(G/H)$

Let $A = k(G/H)$, ring object in $kG-$mod (and beyond) with

$$\mu : A \otimes A \longrightarrow A \qquad \text{given by} \qquad \mu(\gamma \otimes \gamma') = \begin{cases} \gamma & \text{if } \gamma = \gamma' \\ 0 & \text{otherwise.} \end{cases}$$

Theorem

Let $\mathcal{K} = kG-\mathrm{Mod}$. *There is an equivalence* $A-\mathrm{Mod}_{\mathcal{K}} \cong kH-\mathrm{Mod}$ *such that* F_A *becomes* Res^G_H *and* U_A *becomes* CoInd^G_H.

Theorem

Let $\mathcal{K} = kG-\mathrm{stab}$. *There is an equivalence* $A-\mathrm{Mod}_{\mathcal{K}} \cong kH-\mathrm{stab}$, *with* $F_A \cong \mathrm{Res}^G_H$ *and* $U_A \cong \mathrm{CoInd}^G_H$.

Modules and triangulation

Natural question

For $\mathcal{K}$ triangulated, is $M-\mathrm{Mod}_{\mathcal{K}}$ triangulated with $\begin{tikzcd} \mathcal{K} \arrow[d, shift right, "F_M"'] \\ M-\mathrm{Mod}_{\mathcal{K}} \arrow[u, shift right, "U_M"'] \end{tikzcd}$ exact?

Tempting...

$$\begin{tikzcd}
M(x) \arrow[r] \arrow[d, "\varrho"'] & M(x') \arrow[r] \arrow[d, "\varrho'"'] & M(x'') \arrow[r] \arrow[d, dotted, "\varrho''"'] & M(\Sigma x) \arrow[d, "\Sigma\varrho"'] \\
x \arrow[r, "f"] & x' \arrow[r] & x'' \arrow[r] & \Sigma x
\end{tikzcd}$$

Separability – the definition

Ab absurdo

Let (A, μ, η) be a ring object in $\mathcal{K}$. Suppose A–$\mathrm{Bimod}_{\mathcal{K}}$ triangulated with $U : A\text{–}\mathrm{Bimod}_{\mathcal{K}} \longrightarrow \mathcal{K}$ exact. Contemplate an exact triangle

$$A \otimes A \xrightarrow{\mu} A \xrightarrow{\nu} B \longrightarrow \Sigma(A \otimes A)$$

in A–$\mathrm{Bimod}_{\mathcal{K}}$. In $\mathcal{K}$, μ has a section $(\eta \otimes 1)$ hence $\nu = 0$. This forces $\mu : A \otimes A \to A$ to have a section as A-bimodule, *i.e.* A to be *separable*.

Definition

A monad M is *separable* is $\mu : M^2 \to M$ has a section as M-bimodule.

Example

If A is a separable *flat* R-algebra, then A is separable in $\mathrm{D}(R-\mathrm{Mod})$.

Separability – the theorem

Theorem (B. 2010)

M separable exact monad on $\mathcal{K}$ idempotent-complete triangulated, then

$M\text{-Mod}_{\mathcal{K}}$ is triangulated with F_M and U_M both exact: $$\begin{tikzcd} \mathcal{K} \arrow[d, shift right, "F_M"'] \\ M\text{-Mod}_{\mathcal{K}} \arrow[u, shift right, "U_M"'] \end{tikzcd}$$.

Caveat

Strictly speaking, the above is for *pre*-triangulated. For the Octahedron Axiom, we need "distinguished" octahedra. Etc, with higher octahedra: existence and morphism axioms but for n-octahedra, $n \geq 2$.

Example

For $\mathcal{K} = \mathrm{D}(R\text{-Mod})$ and A a separable flat R-algebra, we have $A\text{-Mod}_{\mathcal{K}} \cong \mathrm{D}(A\text{-Mod})$.

1. Motivation

2. Modules in triangulated categories

3. Descent in triangulated categories

Descent data (Grothendieck, Knus-Ojanguren, Cipolla)

Let $M = (M, \mu, \eta)$ be a monad on $\mathcal{K}$ (*e.g.* $M = A \otimes -$). A *descent datum* is triple (x, ϱ, δ) with $x \in \mathcal{K}$ and $\varrho : M(x) \to x$ and $\delta : x \to M(x)$ s.t. :

$$\begin{tikzcd}
M^2(x) \arrow[r, "M(\varrho)"] \arrow[d, "\mu_x"'] \arrow[dr, phantom, "①"] & M(x) \arrow[r, "M(\delta)"] \arrow[d, "\varrho"] \arrow[dr, phantom, "③"] & M^2(x) \arrow[d, "\mu_x"] \\
M(x) \arrow[r, "\varrho"] \arrow[dr, phantom, "②" near start] & x \arrow[r, "\delta"] \arrow[d, "\delta"] \arrow[dl, equal] \arrow[dr, phantom, "④"] & M(x) \arrow[d, "M(\eta_x)"] \\
x \arrow[u, "\eta_x"] & M(x) \arrow[l, "\varrho"] \arrow[r, "M(\delta)"'] \arrow[ul, phantom, "⑤" near start] & M^2(x)
\end{tikzcd}$$

commutes .

Let $\mathrm{Desc}_{\mathcal{K}}(M)$ the *descent category* of such triples. Comparison functor $Q : \mathcal{K} \longrightarrow \mathrm{Desc}_{\mathcal{K}}(M)$ defined by $Q(y) = \big(M(y)\,,\ \mu_y\,,\ M(\eta_y)\big)$.

Descent and triangulated categories

Definition: Effective descent

M satisfies effective descent in $\mathcal{K}$ if $Q : \mathcal{K} \longrightarrow \mathrm{Desc}_{\mathcal{K}}(M)$ is an equivalence.

Obvious necessary condition

M must be faithful since $Q(-) = \big(M(-)\,,\,\ldots\,,\,\ldots\big)$.

A counter-example

Let k be a field and $\mathcal{A}$ the abelian category of (fin. dim.) k-representations of the quiver A_2, or equivalently $\mathcal{A} = \left(\begin{smallmatrix} k & k \\ 0 & k \end{smallmatrix}\right) - \mathrm{mod}$. Objects are arrows $\big[V_1 \xrightarrow{m} V_2\big]$ and morphisms are squares. For each $V \in k-\mathrm{mod}$, we have

$$M_1(V) = \big[V \longrightarrow 0\big]\,, \quad M_2(V) = \big[0 \longrightarrow V\big], \quad M_{12}(V) = \big[V \xrightarrow{\mathrm{id}} V\big]\,.$$

Let $L = \Big[k^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)} k^2\Big] = M_{12}(k) \oplus M_2(k) \oplus M_1(k)$.

Theorem (B. 2010)

$$\begin{tikzcd} \mathrm{D^b}(\mathcal{A}) \arrow[d, shift right, "F := \mathrm{RHom}_{\mathcal{A}}(L\,,\,-)"'] \\ \mathrm{D^b}(k) \arrow[u, shift right, "G := M_1 \oplus M_{12} \oplus M_2[1]"'] \end{tikzcd}$$

Monad $M = G \circ F$ on $\mathrm{D^b}(\mathcal{A})$ faithful but does not satisfy effective descent

The answer

Theorem (B. 2010)

M faithful monad on idempotent-complete triangulated category $\mathcal{K}$. Then M satisfies effective descent $\mathcal{K} \xrightarrow{\sim} \mathrm{Desc}_{\mathcal{K}}(M)$ if and only if M "detects semi-simplicity", i.e. a morphism g in $\mathcal{K}$ has a kernel if $M(g)$ has one.

Remark

Here, g has a kernel $\iff$ g factors as (split mono) $\circ$ (split epi).

Corollary (B. 2010)

A ring object A in an idempotent-complete $\otimes$-triangulated $\mathcal{K}$ satisfies effective descent if and only if A is faithful !

Recall the examples

$A = k(G/H)$

For $H < G$ of finite index, $k(G/H)$ is faithful if $[G : H] \in k^\times$.
On the other hand, it is always separable.

$A = \oplus_{i \in \mathbb{Z}/n} u^{\otimes i}$

For $[u] \in \mathrm{Pic}(\mathcal{K})$ with $\sigma_{u,u} = \mathrm{id}$ and $u^{\otimes n} \simeq 1$, the ring object
$A = 1 \oplus u \oplus \cdots \oplus u^{\otimes(n-1)}$ is always faithful.
It is separable if n is invertible in $\mathcal{K}$.

To be continued...